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ALMOST PERIODIC FUNCTIONS

By

HARALD BOHR

Text translated by
HARVEY COHN

Appendices translated by
F. STEINHARDT

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INTRODUCTION

1. In planning the lectures which form the content of this book I had to make a choice which at first confronted me with a certain difficulty. Since the amount of time in which I had to give the lectures was rather limited, I had to decide in advance whether I wanted to give a more survey-like report on the whole theory or whether I should present only one part – but an essential one – of the theory, this, however, in fullest detail and with complete proofs. I decided for the latter alternative, holding the opinion that the lectures will be more permanently profitable for those attending if they are given the opportunity to assimilate the subject matter without haste – even if this means that the amount of material covered has to be somewhat reduced. The reduction of material was done mainly by dealing only with functions of a _real_ variable, and besides only with _continuous_ functions throughout.

2. In order to make up to some extent for the aforementioned reduction of material I have added two appendices to the lectures proper, as has already been said in the preface. The first of these deals with the _generalization_ of the theory of almost periodic functions (along the lines of Lebesgue's theory of the integral), while the second gives an account of the theory of almost periodic functions of a _complex_ variable. This latter theory by the way, which has grown out of the theory of Dirichlet's series, formed the author's original starting point for the entire investigation.

For the reader who wishes to go deeper into the theory there is a short bibliography at the end, listing some of the pertinent works on the subject.

3. Before starting with the presentation of the theory proper we make some preliminary remarks concerning the problems taken up in the theory of almost periodic functions. Later on, of course, there will be given a detailed and precise formulation of these problems. Generally speaking we can say that the main problem of the theory consists in finding those functions $f(x)$ of the real variable x which are defined for $-\infty < x < \infty$ and which can be resolved into _pure_

<u>vibrations</u>. This statement contains several words whose
meanings must first of all be defined. What are "pure vib-
rations," and what is to be understood by the word "resolv-
ed?" At this place I shall not go any deeper into the con-
cept of resolution, which is most closely connected with the
whole structure of the theory. However, I shall now explain
precisely what is meant by a pure vibration.

     4. As long as only real functions are considered, the
term "pure vibration" applies to such functions as $\cos x$,
$\sin x$ or more generally $\alpha \cos x + \beta \sin x$; these functions are
periodic of period 2π; if arbitrary periods are admitted,
then "pure vibration" will stand for a function of the form
$\alpha \cos \lambda x + \beta \sin \lambda x$. In the following it will be more conven-
ient for formal reasons to consider not real, but complex
functions of the real variable x. By a <u>pure vibration</u> is
then meant any function of the form $a e^{i\lambda x} = a(\cos \lambda x + i \sin \lambda x)$,
where a denotes an arbitrary complex number and λ an arbi-
trary real number. If a is written in the form $a = |a| e^{iv}$,
the function takes on the form $|a| e^{iv} e^{i\lambda x}$. Here $|a|$ gives the
<u>amplitude</u>, v the <u>phase</u> and λ the <u>frequency</u> of the vibration.
(The period of the vibration is (for $\lambda \neq 0$) $2\pi/|\lambda|$). Of
these numbers, $|a|$ and λ are the most important; frequently
$|a|^2$ is considered instead of $|a|$ itself.

     5. The theory of almost periodic functions now deals
with the following problem: <u>Which functions</u> $f(x)$ <u>can be re-</u>
<u>solved into pure vibrations on</u> $-\infty < x < \infty$ <u>i.e., are "re-</u>
<u>presentable" by a trigonometric series of the form</u> $\sum A_n e^{i\Lambda_n x}$?
(We are here concerned only with series having at most denu-
merably many elements, or physically speaking, with functions
whose "spectrum" is a pure line spectrum; the theory of the
Fourier integral $\int_{-\infty}^{\infty} a(\lambda) e^{i\lambda x} d\lambda$, i.e., of the continuous spectra,
is, therefore not, on principle, within the scope of our prob-
lem).
     In the classical case where only <u>harmonic</u> vibrations
are considered, i.e., vibrations of the form $a e^{in\alpha x}, n = 0$,
$\pm 1, \pm 2, \ldots$ the corresponding question leads to the theory of
the ordinary trigonometric series $\sum_{-\infty}^{\infty} a_n e^{in\alpha x}$; any periodic
function of period $p = 2\pi/|\alpha|$ can, as is well known, be dev-
eloped into such a series, the Fourier series of the func-

tion. The essential difference between this case and the a-
bove case where arbitrary vibrations $ae^{i\lambda x}, -\infty < \lambda < \infty$ were
considered, lies in the fact that in the latter case the set
of the frequencies which occur is <u>non-denumerable</u>, whereas
in the first case we are from the very start dealing only
with denumerably many frequencies.

In the lectures are divided into two parts. In the <u>first
part</u> we take up the classical case of harmonic vibrations
$ae^{in\alpha x}$, choosing $\alpha = 1$ for simplicity, so that we are deal-
ing with the theory of ordinary Fourier series of period 2π
This part is essentially of an introductory character and is
intended mainly to make for an easier and deeper understand-
ing of the following part; because of this, the exposition
differs from the usual one in some respects.

In the <u>second part</u> there follows a treatment of the ar-
bitrary vibrations $ae^{i\lambda x}$, which leads to the general theory of
almost periodic functions and of their Fourier series.

6. Bohl's theory of "functions periodic in the more
general sense" occupies an intermediate place between the
theory of periodic and the theory of almost periodic func-
tions. This theory corresponds to the case where only vibr-
ations of the form $ae^{i(n_1\alpha_1 + n_2\alpha_2 + \cdots + n_m\alpha_m)x}$ are considered, where
$\alpha_1, \alpha_2, \ldots, \alpha_m$ are given linearly independent constants while
$n_1, n_2, \ldots, n_m$ run independently through all integers $0, \pm 1,$
$\pm 2, \ldots$ Every function periodic in the more general sense
with the "periods" $p_1 = 2\pi/|\alpha_1|, \; p_2 = 2\pi/|\alpha_2|, \ldots, p_m = 2\pi/|\alpha_m|$
can be developed into a Fourier series of the form
$$\sum a_{n_1, n_2, \ldots, n_m} e^{i(n_1\alpha_1 + n_2\alpha_2 + \cdots + n_m\alpha_m)x}$$
The beautiful investigations by Bohl are closely related to
the theory of almost periodic functions at many points; in a
certain sense, however, these Bohl functions are nearer to
the pure periodic functions than to the almost periodic ones,
the reason being essentially that here, again, the basic sys-
tem of frequencies is denumerable.

7. In the course of building up the theory, there pre-
sented itself, besides the Bohl functions just mentioned,
another simple and important subclass, viz., the class of
<u>limit-periodic</u> functions: These are the functions which can
be developed into trigonometric series of the form $\sum a_r e^{irax}$
where r assumes all rational values. For the sake of brev-

ity, however, I could not in the present lectures enter into
a discussion of either Bohl's functions or the limit-per-
iodic functions.

Also of necessity entirely omitted from the present
publication are investigations such as those by Franklin and
Bochner on almost periodic functions of several – even of
denumerably many – variables; the investigations on the prob-
lem of distribution of values, a problem which has been
attacked from different directions by Wintner and Jessen;
also the investigations of differential equations with al-
most periodic coefficients, carried out mainly by Favard
(who in this followed Bohl).

8. The theory of almost periodic functions was devel-
oped in its main features by the author in three rather long
papers in the Acta Mathematica (Volumes 45, 46 and 47) under
the common title "Zur Theorie der Fast periodische Funktion
en"; the first of these deals with the almost periodic func-
tions of a real variable, while the third takes up the case
of a complex variable.

In building the theory there was encountered a parti-
cular difficulty of intrinsic character; this is the deci-
sion on "completeness" of the system of all functions $e^{i\lambda x}$.
The original proof that this system is indeed a complete one
(in a sense to be made precise later) was very complicated
and involved many conclusions; its guiding idea, however,
was a rather simple one; it was the idea, generally speaking,
that the set of all purely periodic functions (with arbi-
trary periods) can be considered in a certain sense as "being
everywhere dense" within the general class of the almost per-
iodic functions, so that from the very start it appeared
quite possible to derive the validity of the completeness
theorem by means of a limiting process from its well known
validity for purely periodic functions (of a given period,
i.e., for the system of harmonic vibrations $e^{in\alpha x}$).

Wiener succeeded in finding a new proof of this funda-
mental theorem, much shorter than mine but, on the other
hand, using results of the Lebesgue integral theory and the
theory of Fourier integrals; whereas the original proof was
worked by quite elementary means. This proof by Wiener ac-
quired special interest by forming an important starting
point for his beautiful and promising investigations on com-
bined Fourier series and Fourier integrals.

An important and interesting viewpoint which also leads
to a new proof of the fundamental theorem was introduced by
Weyl in his discovery of the connections between the theory
of almost periodic functions and the theory of integral equa-
tions, or rather, mean value equations. The use of consid-
erations from group theory is also characteristic of Weyl's
method, which was later simplified somewhat by Hammerstein.

The proof which I shall present in these lectures is,
however, neither Wiener's nor Weyl's, but a third one due to
De La Vallee Poussin which is close to mine in principle -
although essential use is also made of ideas due to Weyl -
but which is incomparably simpler and can hardly be improved
on in its brevity and elegance. This proof by De La Vallée
Poussin, too, starts from the validity of the completeness
theorem for pure periodic functions, which theorem is easy
to prove and is discussed at length in the first part of
these lectures.

9. Although the fundamental theorem constitutes the
decisive theorem of the whole theory, it should not be con-
sidered the "main theorem" proper; rather, the main theorem
is the so-called approximation theorem - the analogue of
Weierstrass' classical approximation theorem for pure per-
iodic functions, - which gives the very characteristic pro-
perty of the almost periodic functions - that of resolvabil-
ity:

The extant proofs of this approximation theorem are all
based in principle on the fundamental theorem. Since my
original proof (which is based on a relation   essential for
certain problems between the theories of almost periodic and
of limit-periodic functions of infinitely many variables)
there have also here been obtained remarkable simplifications
and new developments, mainly by Bochner and Weyl.

In the lectures I have followed Bochner's proof which
is outstanding as a beautiful and natural generalization of
the Fejer summability method for Fourier series of purely
periodic functions. (This latter is dealt with in the first
part in detail).

10. In closing I add a few remarks on certain details
of the present exposition.

The rather deep proof of the theorem on the integration

of an almost periodic function derives from Bohl; in fact it was possible to carry over this proof - as well as the theorem itself - almost literally from the Bohl functions to the general case of almost periodic functions.

The very last theorem of the entire series of lectures which expresses a particular simplicity in the behavior of a Fourier series $\sum A_n e^{i \Lambda_n x}$ with <u>linearly independent exponents</u> Λ_n, was proved originally by using theorems on Diophantine approximations. The extremely simple proof presented here, which is based on ideas of Bochner and Szidon, I owe to a kind communication from Mr. Fekete.

PURELY PERIODIC FUNCTIONS

AND THEIR FOURIER SERIES

GENERAL ORTHOGONAL SYSTEMS

11. Let $\varphi(x)$ and $\psi(x)$ be two continuous non-vanishing functions defined in the interval $a \leqq x \leqq b$.

When the functions $\varphi(x)$ and $\psi(x)$ are both real, they will be said to be <u>orthogonal to each other</u> if

$$\int_a^b \varphi(x)\,\psi(x)\,dx = 0 .$$

Thus, for example, if m and n are positive integers and $m \neq n$, the two functions $\cos mx$ and $\cos nx$ are orthogonal to each other in $0 \leqq x \leqq 2\pi$, since

$$\int_0^{2\pi} \cos mx \cos nx\,dx = \int_0^{2\pi} \tfrac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} = 0 .$$

When $\varphi(x)$ and $\psi(x)$ are complex functions, they will be said to be orthogonal to each other in $a \leqq x \leqq b$ if,

$$\int_a^b \varphi(x)\,\overline{\psi(x)}\,dx = 0 \quad \left(\text{or} \quad \int_a^b \overline{\varphi(x)}\,\psi(x)\,dx = 0 \right).$$

Here a horizontal stroke above the quantity changes it to its complex conjugate value. Thus, if m and n denote arbitrary integers and $m \neq n$, the two functions e^{imx} and e^{inx} are orthogonal to each other in $0 \leqq x \leqq 2\pi$, since

$$\int_0^{2\pi} e^{imx} e^{-inx}\,dx = \int_0^{2\pi} e^{i(m-n)x}\,dx = \left[\frac{e^{i(m-n)x}}{i(m-n)}\right]_0^{2\pi} = 0 .$$

In the two examples just cited the functions considered are periodic, with period 2π. From the orthogonality property in the interval $0 \leqq x \leqq 2\pi$ we can deduce orthogonality in an arbitrary interval $a \leqq x \leqq a + 2\pi$ of length 2π.

In what follows, the functions under consideration are <u>complex</u> unless we specifically state the contrary.

12.   It is often convenient to consider in place of the integral $\int\limits_a^b \cdots$ the meanvalue $\dfrac{1}{b-a}\int\limits_a^b \cdots$ Letting $\varphi(x)$ be any continuous function defined $a \leqq x \leqq b$ we make use of the notation

$$\frac{1}{b-a}\int\limits_a^b \varphi(x)\,dx = M\{\varphi(x)\} = M\{\varphi\}.$$

The function $\varphi(x)$ will be said to be <u>normalized</u> (in the interval $a \leqq x \leqq b$) if the mean value is unity, or, if

$$\frac{1}{b-a}\int\limits_a^b \varphi(x)\,\overline{\varphi(x)}\,dx = \frac{1}{b-a}\int\limits_a^b |\varphi(x)|^2\,dx = 1.$$

Thus, for example, if n is an integer, the function e^{inx} is normalized in $0 \leqq x \leqq 2\pi$, and in fact even in any interval $a \leqq x \leqq b$, since

$$\int\limits_a^b e^{inx}\,e^{-inx}\,dx = \int\limits_a^b 1\,dx = b-a.$$

On the other hand, the function $\cos nx$, where n denotes a positive integer,   is not normalized in $0 \leqq x \leqq 2\pi$, since

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \cos^2 nx\,dx = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1+\cos 2nx}{2}\,dx = \frac{1}{4\pi}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \frac{1}{2}.$$

Suppose the function $\varphi(x)$ is not normalized.   Then, (if it does not vanish identically), it always can be transformed into a normalized function by multiplication by a suitable constant.   In fact, the mean value

$$\frac{1}{b-a}\int\limits_a^b |\varphi(x)|^2\,dx = M\{|\varphi|^2\}$$

is always real and   in this case greater than zero.   Consequently, the function
$$\frac{\varphi(x)}{\sqrt{M\{|\varphi|^2\}}}$$
has meaning and is obviously normalized.   Thus, for example, if n denotes a positive integer, $\sqrt{2}\cos nx$ is a normalized fvnction in $0 \leqq x \leqq 2\pi$ and more generally in any interval $a \leqq x \leqq a + 2\pi$.

13.   A (finite or infinite) set of functions continuous in $a \leqq x \leqq b$ $\varphi(x), \quad \psi(x), \quad \ldots$
is called an <u>orthogonal system</u>, if every two of the func-

tions are orthogonal to each other. The set will be called
__normal__, if each function of the system is normalized in the
sense of the last article.

A __normal orthogonal system__ is, therefore, a system of
functions for which the relation

$$M\{\varphi \overline{\psi}\} = 0$$

holds for any two distinct functions of the system and for
which the relation

$$M\{\varphi \overline{\varphi}\} = M\{|\varphi|^2\} = 1$$

holds for any single function of the system.

__Example.__ The system of functions

$$e^{inx}, \qquad\qquad n = 0, \pm 1, \pm 2, \ldots$$

is a normal orthogonal system in $0 \leqq x \leqq 2\pi$ or more generally
in each interval $a \leqq x \leqq a + 2\pi$ of length 2π. In fact,

$$M\{e^{inx} e^{-imx}\} = 0 \text{ for } n \neq m$$

and

$$M\{e^{inx} e^{-inx}\} = M\{1\} = 1$$

hold for each n. Each sub-set of the system under consid-
eration is obviously itself a normal orthogonal system. The
system as a whole, however, is __complete__ (__vollstaendig__), in
the sense that it is not a sub-system of a larger normal
orthogonal system. This important property will be analyzed
later in some detail.

FOURIER CONSTANTS WITH RESPECT TO A NORMALIZED ORTHOGONAL
SYSTEM. THEIR MINIMAL PROPERTY. BESSEL'S FORMULA.BESSEL'S
INEQUALITY.

14. Let $\{\varphi(x)\}$ denote a normal orthogonal system in the
interval $a \leqq x \leqq b$ and $F(x) = U(x) + iV(x)$ an arbitrary func-
tion continuous in $a \leqq x \leqq b$. We call the number

$$F_\varphi = M\{F(x) \overline{\varphi(x)}\} = \frac{1}{b-a} \int_a^b F(x) \overline{\varphi(x)} \, dx .$$

the __Fourier constant__ of $F(x)$ with respect to a function $\varphi(x)$
of an orthogonal system. Now, choosing a certain finite
number of functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$, of the system $\{\varphi(x)\}$
and forming the sum

$$S(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x) + \cdots + c_n \varphi_n(x) .$$

where $c_1, c_2, \ldots, c_n$ are complex constants, we propose the fol-
lowing

Problem. The constants $c_1, c_2, \ldots, c_n$ are to be so chosen that the sum $S(x)$ approximates the given function $F(x)$ in the best possible manner in the sense of <u>the method of least squares</u>. In other words the mean value

$$M\{|F(x) - S(x)|^2\} = \frac{1}{b-a}\int_a^b |F(x) - S(x)|^2 dx$$

is to be as small as possible.

It will be shown that this problem has one and only one solution.

Through direct calculation and application of the identity $|A|^2 = A\bar{A})$, we obtain

$$M\{|F - S|^2\} = M\left\{\left(F(x) - \sum_{\nu=1}^n c_\nu\, \varphi_\nu(x)\right)\left(\overline{F(x)} - \sum_{\nu=1}^n \bar{c}_\nu\, \overline{\varphi_\nu(x)}\right)\right\}$$

$$= M\{F\bar{F}\} - \sum_{\nu=1}^n \bar{c}_\nu M\{F\overline{\varphi_\nu}\} - \sum_{\nu=1}^n c_\nu M\{\bar{F}\varphi_\nu\}$$

$$+ \sum_{\nu_1=1}^n \sum_{\nu_2=1}^n c_{\nu_1} \bar{c}_{\nu_2} M\{\varphi_{\nu_1} \overline{\varphi_{\nu_2}}\},$$

and thus, on account of

$$M\{\varphi_{\nu_1}\overline{\varphi_{\nu_2}}\} = \begin{cases} 0 \text{ for } \nu_1 \neq \nu_2 \\ 1 \text{ for } \nu_1 = \nu_2, \end{cases}$$

$$M\{|F - S|^2\} = M\{|F|^2\} - \sum_1^n \bar{c}_\nu F_{\varphi_\nu} - \sum_1^n c_\nu \overline{F_{\varphi_\nu}} + \sum_{\nu=1}^n c_\nu \bar{c}_\nu$$

$$= M\{|F|^2\} + \sum_1^n (c_\nu - F_{\varphi_\nu})(\bar{c}_\nu - \overline{F_{\varphi_\nu}}) - \sum_1^n F_{\varphi_\nu}\overline{F_{\varphi_\nu}}.$$

We thus obtain the simple formula

$$M\{|F(x) - S(x)|^2\} = M\{|F(x)|^2\} - \sum_1^n |F_{\varphi_\nu}|^2 + \sum_1^n |c_\nu - F_{\varphi_\nu}|^2.$$

This formula gives us the solution of the proposed problem, by showing us that to make the mean value

$$M\{|F(x) - S(x)|^2\}$$

as small as possible, we must choose the constants $c_1, c_2, \ldots, c_n$ as the Fourier constants $F_{\varphi_1}, F_{\varphi_2}, \ldots, F_{\varphi_n}$. In this case only, does the last term of the formula (the one in which the constants c_ν appear) assume its smallest possible value, namely, zero.

This <u>minimal property</u> shows the meaning of the Fourier constants. It should be noted that the solution of the prob-

lem is unique and that the value which must be given to each constant c_ν depends only on the corresponding function $\varphi_\nu(x)$ and not on any other functions of the system.

15. If in the above result the Fourier constant c_ν is substituted for the constant F_{φ_ν}, we obtain Bessel's formula

$$M\left\{\left|F(x) - \sum_1^n F_{\varphi_\nu}\,\varphi_\nu(x)\right|^2\right\} = M\{|F(x)|^2\} - \sum_1^n |F_{\varphi_\nu}|^2\,.$$

Since the left hand member of this formula is obviously ≥ 0 we obtain Bessel's inequality as a corollary:

$$\sum_1^n |F_{\varphi_\nu}|^2 \leq M\{|F(x)|^2\}\,.$$

This is also valid for an arbitrary number of functions $\varphi_\nu(x)$ selected from our system.

FOURIER SERIES OF PERIODIC FUNCTIONS

16. We now proceed to make a closer study of the most important of all orthogonal systems, namely the previously mentioned normal orthogonal system

$$e^{inx}, \qquad\qquad n = 0, \pm 1, \pm 2 \ldots$$

in the interval $0 \leq x \leq 2\pi$.

We let $P(x)$ denote an arbitrary continuous function of period 2π. (Later on we shall use the notation $P_1(x), \ldots$). We represent the n th Fourier coefficient of $P(x)$ by

$$a_n = M\{P(x)\,e^{-inx}\} = \frac{1}{2\pi}\int_0^{2\pi} P(x)\,e^{-inx}\,dx\,,$$

and we call the trigonometric series $\sum_{-\infty}^{\infty} a_n e^{inx}$ formed with the coefficients a_n, the Fourier series of $P(x)$. we use for this the abbreviation

$$P(x) \sim \sum a_n e^{inx}\,.$$

We stress the fact that up to now we have been dealing with a purely formal matter. We can say nothing at present of the arrangement of the terms, or of the convergence of the series, etc.

17. To justify the selection of these numbers a_n as coefficients, we point out the following:

1. According to the minimal property (§14), every

finite sum $\sum * a_n e^{inx}$ of terms of the Fourier series approximates the function $P(x)$ better _in the mean_ than any other linear combination $\sum * c_n e^{inx}$ of the same functions e^{inx} . This means that

$$M\{|P(x) - \sum * a_n e^{inx}|^2\} < M\{|P(x) - \sum * c_n e^{inx}|^2\},$$

as long as it is not true that $c_n = a_n$ for all values of n involved.

From now on, the symbol $\sum * \ldots$ will always denote a finite sum.

2. If we make the formal substitution,

$$P(x) = \sum_{-\infty}^{\infty} b_n e^{inx}$$

then, after multiplying by e^{-inx} and forming mean values, we obtain simply

$$M\{P(x) e^{-inx}\} = b_n .$$

There is one important case where this formal type of calculation is clearly permissible, namely in the following

Special Theorem. If $P(x)$ can be developed into a trigonometric series $\sum b_n e^{inx}$, which converges _uniformly for all_ x (regardless of the order of the terms) then this series must be none other than the Fourier series of $P(x)$.

In this case the series resulting from the multiplication by e^{-inx} also converges uniformly. The term by term integration is, therefore, permissible and it yields $a_n = b_n$ for each n.

With later applications in mind, we restate our theorem in the following form:

If a trigonometric series $\sum b_n e^{inx}$ converges uniformly (regardless of the order of its terms) with $S(x)$, as its sum, then it is precisely the Fourier Series of the sum $S(x)$.

For example, a "trigonometric polynomial" $\sum * b_n e^{inx}$ is the Fourier Series of its sum $S(x)$.

OPERATIONS WITH FOURIER SERIES

18. We are now going to examine the connection between the functions $P(x)$ and their Fourier series -- at first from a rather _formal_ standpoint. We wish to show, for instance, that simple operations with functions $P(x)$ are paralleled by

the corresponding formal operations with their Fourier series.
From $P(x) \sim \sum a_n e^{inx}$, the simple formula

(1) $$kP(x) \sim \sum k a_n \cdot e^{inx},$$

is seen immediately to follow, where k denotes an arbitrary complex constant. In fact,

$$M\{kP(x)\, e^{-inx}\} = kM\{P(x)\, e^{-inx}\} = k a_n.$$

(2) $$e^{imx} P(x) \sim \sum a_n e^{i(m+n)x}, \text{ i.e., } \sum a_{n-m} e^{inx}.$$

For $$M\{e^{imx} P(x)\, e^{-inx}\} = M\{P(x)\, e^{-i(n-m)x}\} = a_{n-m}.$$

(3) $$P(x+k) \sim \sum a_n e^{in(x+k)}, \text{ i.e., } \sum a_n e^{ink} \cdot e^{inx},$$

where k denotes an arbitrary real constant. For

$$M\{P(x+k)\, e^{-inx}\} = e^{ink} M\{P(x+k)\, e^{-in(x+k)}\},$$

Hence, if we set $x + k = x'$

$$= e^{ink} M\{P(x')\, e^{-inx'}\} = e^{ink} \cdot a_n.$$

(4) $$\overline{P(x)} \sim \sum \overline{a_n}\, e^{-inx}, \text{ i.e., } \sum \overline{a_{-n}}\, e^{inx}$$

since $$M\{\overline{P(x)}\, e^{-inx}\} = \overline{M\{P(x)\, e^{inx}\}} = \overline{a_{-n}}.$$

From $P_1(x) \sim \sum a_n e^{inx}$ and $P_2(x) \sim \sum b_n e^{inx}$ it further follows that

(5) $$P_1(x) + P_2(x) \sim \sum (a_n + b_n) e^{inx}.$$

since

$$M\{(P_1(x) + P_2(x))\, e^{-inx}\} = M\{P_1(x)\, e^{-inx}\} + M\{P_2(x)\, e^{-inx}\} = a_n + b_n$$

(From the formulas (1) and (5) it follows, for example, that the Fourier series of the difference $P_1(x) - P_2(x)$ results from the formal subtraction of the Fourier series of $P_1(x)$ and $P_2(x)$.)

19. From two of the above formulas, namely (1) and (2), we arrive at the formula for the multiplication of an arbitrary periodic function $P(x)$ by a rather special function. The following <u>general multiplication theorem</u> is really deeper than the two preceding theorems:

(6) $$P_1(x)\, P_2(x) \sim \sum c_n e^{inx} \text{ with } c_n = \sum_{\mu+\nu=n} a_\mu b_\nu.$$

These formulas can first be proved, however, only later (in §§ 27-29.)·

20. It is simple, and yet important for later applications, to find the Fourier series of a so called <u>folded function</u> (<u>gefalteten Funktion</u>) or <u>convolution function</u>

$$Q(x) = M_t\{P_1(x+t)\,P_2(t)\} = \frac{1}{2\pi}\int\limits_0^{2\pi} P_1(x+t)\,P_2(t)\,dt\,.$$

This function is obviously another continuous periodic function of period 2π, and the formula

(7) $$Q(x) = M_t\{P_1(x+t)\,P_2(t)\} \sim \sum a_n\,b_{-n}\,e^{inx}$$

is valid. This follows immediately by a simple change of the order of integration. In fact such a procedure yields:

$$
\begin{aligned}
M_x\{Q(x)\,e^{-inx}\} &= M_x M_t\{M_t\{P_1(x+t)\,P_2(t)\}\,e^{-inx}\} \\
&= M_t\{P_2(t)\,M_x\{P_1(x+t)\,e^{-inx}\}\} = M_t\{P_2(t)\,a_n\,e^{int}\} \\
&= a_n\,M_t\{P_2(t)\,e^{int}\} = a_n\,b_{-n}\,.
\end{aligned}
$$

A certain special case is rather important. For $P_1(x) = P(x) \sim \sum a_n e^{inx}$ and $P_2(x) = \overline{P(x)} \sim \sum \overline{a_{-n}}\,e^{inx}$ this formula gives:

(8) $$Q(x) = M_t\{P(x+t)\,\overline{P(t)}\} \sim \sum a_n\,\overline{a_{+n}}\,e^{inx},\ \text{i.e.,}\ \sum|a_n|^2\,e^{inx}\,.$$

The two formulas (7) and (8) for setting up the Fourier series for the functions $F(t) = P_1(x+t)P_2(t)$ and $P(x+t)\,\overline{P(t)}$ respectively are particularly interesting for the reason that a formal derivation of the multiplication theorem is desired. We might have expected the Formulas (7) and (8) to be as deep as (6). Yet we have just seen that this is not the case. We shall later give a proof of the multiplication theorem just on the basis of formula (7).

21. If a given sequence of periodic functions

$$P_m(x) \sim \sum a_n^{(m)}\,e^{inx}$$

converges uniformly for all x towards a limit function $P(x)$ as $m \to \infty$ then

(9) $$P(x) = \lim_{m \to \infty} P_m(x) \sim \lim_{m \to \infty} \sum a_n^{(m)}\,e^{inx}\,.$$

By this we mean that if $P(x) \sim \sum a_n e^{inx}$, then for each n the following formula is valid

$$a_n = \lim_{m \to \infty} a_n^{(m)}$$

and indeed, uniformly with respect to n.

The proof follows immediately from the formula

$$a_n - a_n^{(m)} = M\{(P(x) - P_m(x))e^{-inx}\}.$$

If $\varepsilon > 0$ is prescribed arbitrarily, then there exists a $M = M(\varepsilon)$, such that as long as $m > M$,

$$|P(x) - P_m(x)| < \varepsilon$$

for all x. It follows from this, that

$$|a_n - a_n^{(m)}| < M\{\varepsilon \cdot 1\} = \varepsilon$$

for $m > M$ and all n.

22. Let $P(x) \sim \sum a_n e^{inx}$ and let

$$P_1(x) = \int P(x)\,dx$$

be an arbitrary <u>indefinite integral</u> of $P(x)$. For arbitrary x

$$P_1(x + 2\pi) - P_1(x) = \int_{x}^{x+2\pi} P(\xi)\,d\xi = 2\pi a_0,$$

hence $P_1(x)$ will be periodic with period 2π when and only when $a_0 = 0$. Here, the following formula is valid:

$$(10) \qquad P_1(x) \sim C + \sum_{n \neq 0} \frac{a_n}{in} e^{inx}.$$

For, using integration by parts, we obtain the following result for $n \neq 0$

$$M\{P_1(x)e^{-inx}\} = \frac{1}{2\pi}\int_{0}^{2\pi} P_1(x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[P_1(x)\frac{e^{-inx}}{-in}\right]_0^{2\pi} + \frac{1}{in} \cdot \frac{1}{2\pi}\int_{0}^{2\pi} P(x)e^{-inx}\,dx = 0 + \frac{a_n}{in}$$

TWO FUNDAMENTAL THEOREMS. THE UNIQUENESS THEOREM AND PARSEVAL'S EQUATION

23. From the preceding investigations on the relationship between a function $P(x)$ and its Fourier series, the fundamental question arises <u>whether or not a function is uniquely determined by its Fourier series</u>. That is, are the Fourier series corresponding to two distinct functions $P_1(x)$ and $P_2(x)$ distinct? The (affirmative) answer to this question will be given by the <u>Uniqueness theorem</u> given below. At this point

we shall content ourselves with another formulation of the
theorem.   Thus let us take

$$P_1(x) \sim \sum b_n e^{inx} \quad \text{and} \quad P_2(x) \sim \sum c_n e^{inx};$$

then as was previously noticed,

$$P(x) = P_1(x) - P_2(x) \sim \sum (b_n - c_n) e^{inx}.$$

The two functions $P_1(x)$ and $P_2(x)$ then have the same Fourier
series when and only when all the Fourier coefficients of
the function $P(x) = P_1(x) - P_2(x)$ are zero. Henceforth the
uniqueness theorem can also be formulated as follows. There
exists no non-vanishing function $P(x)$ for which all Fourier
constants a_n are equal to zero. In other words, (since each
non-vanishing function $P(x)$ can be normalized by multiplication
by a suitable constant) there exists no function $P(x)$ which
satisfies the following two conditions:

$$M\{P(x) e^{-inx}\} = 0, \qquad n = 0, \pm 1, \pm 2, \ldots,$$
$$M\{|P(x)|^2\} = 1.$$

In this form the uniqueness theorem states simply that the
normal orthogonal system $\{e^{inx}\}$ can not be enlarged by the
addition of a new periodic function of period 2π . Thus the
system is complete.

    24.   Let $P(x) \sim \sum a_n e^{inx}$ again   denote an arbitrary
(continuous) periodic function of period 2π.
    From Bessel's formula

$$M\{|P(x) - \sum{}^* a_n e^{inx}|^2\} = M\{|P(x)|^2\} - \sum{}^* |a_n|^2$$

(where $\sum^* \ldots$ denotes the sum of an arbitrary finite number
of terms) the Bessel Inequality follows:

$$\sum{}^* |a_n|^2 \leqq M\{|P(x)|^2\}$$

Accordingly, the infinite series (with non-negative terms)

$$\sum_{-\infty}^{\infty} |a_n|^2$$

must be convergent and in fact its sum must be

$$\leqq M\{|P(x)|^2\}.$$

(In particular $a_n \to 0$ as $|n| \to \infty$.] .)   We write

$$D = M\{|P(x)|^2\} - \sum_{-\infty}^{\infty} |a_n|^2 \geqq 0 .$$

The question of whether $D > 0$ or $D = 0$ is then of funda-
mental importance.

    1. If $D = 0$, then for each $\varepsilon > 0$ we can choose a fin-
ite sum $\sum^* |a_n|^2$ of terms of the series $\sum_{-\infty}^{\infty} |a_n|^2$ so that

$$M\{|P(x)|^2\} - \sum{}^* |a_n|^2 < \varepsilon ,$$

or

$$M\{|P(x) - \sum{}^* a_n e^{inx}|^2\} < \varepsilon ,$$

and $P(x)$ can accordingly be approximated in the mean by trig-
onometric polynomials to any degree of accuracy.

    2. If, however, $D > 0$, then for each trigonometric
polynomial $\sum^* c_n e^{inx}$ with arbitrary coefficients, we have by
the minimal property of Fourier constants

$$M\{|P(x) - \sum{}^* c_n e^{inx}|^2\} \geqq M\{|P(x) - \sum{}^* a_n e^{inx}|^2\} ;$$

and thus from Bessel's formula, we have the relation,

$$M\{|P(x) - \sum{}^* c_n e^{inx}|^2\} \geqq M\{|P(x)|^2\} - \sum{}^* |a_n|^2$$

$$\geqq M\{|P(x)|^2\} - \sum_{-\infty}^{\infty} |a_n|^2 = D > 0 .$$

Now in this case $P(x)$ can not be approximated in the mean
by trigonometric polynomials with arbitrary accuracy (in
fact not with an error smaller than D).

    We shall bring out in the proof that $D = 0$ always,
i.e., that "Parseval's equation"

$$\sum_{-\infty}^{\infty} |a_n|^2 = M\{|P(x)|^2\}$$

holds for each function $P(x)$. In accordance with what has
just been said this theorem is equivalent to the following;
Every continuous function $P(x)$ of period 2π can be approx
mated in the mean as closely as desired by trigonometric
polynomials $\sum^* c_n e^{inx}$.

    25. Before we prove the two fundamental theorems, we
wish to first discuss the relationship between them. To
this end, we first prove:

1.   The uniqueness theorem follows from Parseval's equation.

For this reason the Parseval equation is also frequently called the "completeness" relation.

Proof: Let $P(x)$ have all its Fourier constants a_n equal to zero; then it follows from the Parseval equation

$$M\{|P(x)|^2\} = \sum_{-\infty}^{\infty} |a_n|^2,$$

that $M\{|P(x)|^2\} = 0$, so that obviously $P(x)$ must vanish identically.

The proof of the converse is just as easy (but less trivial):

2.   Parseval's equation follows from the uniqueness theorem.

Proof: Starting from $P(x) \sim \sum a_n e^{inx}$ we construct the following function (as in 20, formula (8)) :

$$Q(x) = M_t\{P(x+t)\,\overline{P(t)}\} \sim \sum |a_n|^2 e^{inx}.$$

Since the series $\sum |a_n|^2$ is convergent (with sum

$\leqq M\{|P(x)|^2\}$),   the last series $\sum |a_n|^2 e^{inx}$ (arranged in any order) converges uniformly for all x and is accordingly the Fourier series of its sum $S(x)$ The two functions $Q(x)$ and $S(x)$ have the same Fourier series (namely $\sum |a_n|^2 e^{inx}$), thus according to the uniqueness theorem $Q(x) = S(x)$, i.e.,

$$Q(x) = \sum |a_n|^2 e^{inx}.$$

If, in particular, we choose $x = 0$ this equation yields

$$Q(0) = M_t\{P(t)\,\overline{P(t)}\} = M\{|P(t)|^2\} = \sum |a_n|^2$$

Parseval's equation.

LEBESGUE'S PROOF OF THE UNIQUENESS THEOREM

26.   To prove the two fundamental theorems, it suffices to establish the validity of either of them.   We here give a proof of the uniqueness theorem due to Lebesgue.

For the later development of this book it would not have been necessary to give this proof; in fact the two fundamental theorems follow from the Féjer theorem proved below. Moreover, there is no uniqueness proof in the general theory of almost periodic functions that corresponds to the simple Lebesgue proof.

Let a continuous function $P(x)$ of period 2π have all its Fourier coefficients equal to zero, i.e., let

$$M\{P(x)e^{-inx}\} = 0$$

for all n. We are to show that $P(x)$ is identically zero.

It is no limitation of generality to take $P(x)$ real. In fact if all Fourier constants of $P(x)$ are zero the same is true for the conjugate function $\overline{P(x)}$ and hence for the two functions $\frac{1}{2}\left(P(x) + \overline{P(x)}\right)$ and $\frac{1}{2i}\left(P(x) - \overline{P(x)}\right)$, the real and imaginary parts of $P(x)$. These two functions can vanish however only when $P(x)$ is identically zero.

We will prove the theorem indirectly; we assume that $P(x)$ is not identically zero. Suppose, for instance, that $P(x_0) \neq 0$. It is no restriction of generality to take $x_0 = 0$ and $P(0) > 0$; otherwise we replace $P(x)$ by $P(x + x_0)$ or $-P(x + x_0)$. We now choose $d (< \pi)$ so small that $P(x) > c > 0$ for $-d < x < d$. We wish to construct a trigonometric polynomial $Q(x) = \sum^* c_n e^{inx}$ that takes only real values and that places special emphasis on the neighborhood of $x = 0$.

To this end we consider the function

$$\psi(x) = \cos x + (1 - \cos d)$$

Fig. 1.

This function fulfills the following conditions.

1. $\psi(d) = \psi(-d) = 1$.
2. $|\psi(x)| < 1$ for $-\pi < x < -d$ and $d < x < \pi$.
3. $\psi(x) > 1$ for $-d < x < d$ and in particular

$$\psi(x) > g > 1 \quad \text{for} \quad -\frac{d}{2} < x < \frac{d}{2}.$$

We now choose $Q(x) = (\psi(x))^N$, where N denotes any positive integer. Since $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $(\psi(x))^N$ is a trigonometric
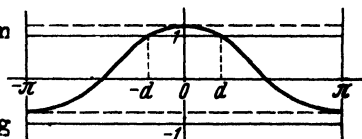
polynomial: $(\psi(x))^N = \sum^* c_n e^{inx}$. It follows, since $P(x)$ is by assumption orthogonal to all functions e^{inx} that $P(x)$ is also orthogonal to $(\psi(x))^N$ or, since both functions are real, that

$$\int_{-\pi}^{\pi} P(x)(\psi(x))^N dx = 0$$

For N sufficiently large, this leads to a contradiction. We denote the upper limit of $|P(x)|$ by K. Then, for every N

$$\int_{-\pi}^{\pi} P(x)(\psi(x))^N dx = \int_{-\pi}^{-d} + \int_{-d}^{-\frac{d}{2}} + \int_{-\frac{d}{2}}^{\frac{d}{2}} + \int_{\frac{d}{2}}^{d} + \int_{d}^{\pi} P(x)(\psi(x))^N dx$$

$$> -(\pi - d)K + 0 + dcg^N + 0 - (\pi - d)K > dcg^N - 2\pi K.$$

In this inequality, however, the right side $dcg^N - 2\pi K$ is surely > 0, for N sufficiently large. (Indeed it approaches $+\infty$ as $N \to \infty$). This contradiction proves the uniqueness theorem.

THE MULTIPLICATION THEOREM

27.   Now that we have proved the uniqueness theorem and Parseval's equation, we shall prove the general multiplication theorem (6) of § 19, and wish, indeed, to show that this theorem is equivalent to the two (mutually equivalent) principal theorems.

The general multiplication theorem states the following: If $P_1(x) \sim \sum a_n e^{inx}$ and $P_2(x) \sim \sum b_n e^{inx}$, then

(6) $$P_1(x)P_2(x) \sim \sum c_n e^{inx} \qquad c_n = \sum_{\mu+\nu=n} a_\mu b_\nu.$$

i.e., for each value of n

$$M\{P_1(x)\,P_2(x)\,e^{-inx}\} = \sum_{\mu+\nu=n} a_\mu b_\nu = \sum_{\mu=-\infty}^{\infty} a_\mu b_{n-\mu}.$$

Let us first notice that for each value of n, the series $\sum_{\mu+\nu=n} a_\mu b_\nu = \sum_{\mu=-\infty}^{\infty} a_\mu b_{n-\mu}$ is absolutely convergent. This follows immediately from the elementary inequality $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$,

when we recall the convergence of the two series $\sum\limits_{\mu=-\infty}^{\infty}|a_\mu|^2$

and $\sum\limits_{\nu=-\infty}^{\infty}|b_\nu|^2 = \sum\limits_{\mu=-\infty}^{\infty}|b_{n-\mu}|^2$.

We further notice that it suffices to consider the special case $n=0$, so that we are now working with the constant term of the Fourier series, or with the equation

(*) $$M\{P_1(x)\,P_2(x)\} = \sum_{\mu+\nu=0} a_\mu b_\nu = \sum_\mu a_\mu b_{-\mu}.$$

For if this equation holds we can simply substitute the

function $P_2(x)e^{-inx} \sim \sum\limits_\mu b_{n+\mu} e^{i\mu x}$ in place of $P_2(x)$, and obtain

the more general equation

$$M\{P_1(x)\,P_2(x)\,e^{-inx}\} = \sum_{\mu=-\infty}^{\infty} a_\mu b_{n-\mu}.$$

28.   We now proceed to the proof itself.

1.   Parseval's equation (and therefore the uniqueness theorem also) follows from the multiplication theorem.

We choose, in particular, $P_2(x) = \overline{P_1(x)} \sim \sum \overline{a_{-n}}\, e^{inx}$, then (*) yields the equation

$$M\{P_1(x)\,\overline{P_1(x)}\} = \sum_\mu a_\mu \overline{a_{+\mu}},$$

which is precisely the Parseval equation

$$M\{|P_1(x)|^2\} = \sum_\mu |a_\mu|^2.$$

2.   The Multiplication theorem follows from uniqueness theorem (and thus from Parseval's equation).

The proof is completely analogous to an earlier proof. According to formula (7) in §20 we construct the function

$$Q(x) = M_t\{P_1(x+t)\,P_2(t)\} \sim \sum_\mu a_\mu b_{-\mu}\, e^{i\mu x}.$$

Since (as we noticed before) the series $\sum\limits_\mu a_\mu b_{-\mu}$ is absolu-

tely convergent, the series $\sum\limits_\mu a_\mu b_{-\mu} e^{i\mu x}$ (in any order of

arrangement of terms) is uniformly convergent and is, there-fore, the Fourier series of its sum $S(x)$. On the other hand, it is the Fourier series of $Q(x)$.   Therefore, it follows from the uniqueness theorem that $Q(x) = S(x)$, or that

$$M_t\{P_1(x+t)\,P_2(t)\} = \sum_\mu a_\mu b_{-\mu} e^{i\mu x}.$$

If we choose $x = 0$, the desired relation (*) follows.

29. By the aid of a trick we shall derive the multiplication theorem directly from the more specialized Parseval equality without using the uniqueness theorem as an intermediary.

We have only to use the elementary identity

$$u v = \tfrac{1}{4}\{|u + \bar{v}|^2 - |u - \bar{v}|^2 + i|u + i\bar{v}|^2 - i|u - i\bar{v}|^2\}$$

If we apply this to the product $P_1(x) P_2(x)$, we get

$$M\{P_1 P_2\}$$
$$= \tfrac{1}{4}[M\{|P_1 + \overline{P_2}|^2\} - M\{|P_1 - \overline{P_2}|^2\} + iM\{|P_1 + i\overline{P_2}|^2\} - iM\{|P_1 - i\overline{P_2}|^2\}].$$

If, further, we apply Parseval's equation to the functions $P_1(x) + \overline{P_2(x)}$, $P_1(x) - \overline{P_2(x)}$, $P_1(x) + i\overline{P_2(x)}$ and $P_1(x) - i\overline{P_2(x)}$ whose coefficients we know from formulas (1), (4) and (5) of § 18, we obtain the result that in the preceding relation the right hand member

$$= \tfrac{1}{4}\left[\sum_{-\infty}^{\infty}|a_n + \overline{b_{-n}}|^2 - \sum_{-\infty}^{\infty}|a_n - \overline{b_{-n}}|^2 + i\sum_{-\infty}^{\infty}|a_n + i\overline{b_{-n}}|^2 - i\sum_{-\infty}^{\infty}|a_n - i\overline{b_{-n}}|^2 \right]$$

Now we can apply the identity again, but in the opposite direction, and obtain

$$M\{P_1 P_2\} = \sum_{-\infty}^{\infty} a_n b_{-n} \, . \, .$$

This is the desired relation (*).

SUMMABILITY OF FOURIER SERIES
FEJER'S THEOREM

30. We have seen that the Fourier Series $\sum a_n e^{inx}$ of a function $P(x)$ uniquely determines this function. This fact is related to the question of whether or not one can actually **compute** the function corresponding to a Fourier Series. To deal with this problem it will first be necessary to choose a particular ordering of the terms. (This was unnecessary for the preceding developments). We shall choose the "natural" order

$$a_0 + (a_1 e^{ix} + a_{-1} e^{-ix}) + \cdots + (a_n e^{inx} + a_{-n} e^{-inx}) + \cdots .$$

As the nth, partial sum of the series, we take the sum

$$s_n(x) = \sum_{-n}^{n} a_\nu e^{i\nu x} .$$

We might next consider **convergence**. This would, however, be unwise; for as Du Bois Reymond first showed, there are

functions whose Fourier series (with terms taken in the order just given) are <u>divergent</u> at certain points. On the other hand we shall obtain excellent results if, with Féjer, we consider, not convergence, but <u>summability</u>.

31. We recall the definition of summability. (By summability we mean merely Cesaro, summability of the first order). Let $\sum_0^\infty u_n$ be a given infinite series and let $s_n = \sum_{\nu=0}^n u_\nu$ be the nth partial sum of the series. If the mean value is formed

$$S_n = \frac{1}{n}(s_0 + s_1 + \cdots + s_{n-1}) = \frac{1}{n}\sum_{\nu=0}^n (n-\nu)u_\nu = \sum_{\nu=0}^n \left(1 - \frac{\nu}{n}\right)u_\nu$$

then the given sum is said to be <u>summable with sum</u> U, if $S_n \to U$ for $n \to \infty$.

Note: Summability is a more general conception than convergence, i.e., if the series $\sum_0^\infty u_n$ is convergent with sum U, then it is also summable with the same sum U. On the other hand there exist series that are summable but not convergent.

32. The Féjer theorem states the following:
<u>The Fourier series</u>

$$\sum_0^\infty u_n = a_0 + (a_1 e^{ix} + a_{-1}e^{-ix}) + \cdots + (a_n e^{inx} + a_{-n}e^{-inx}) + \cdots$$

<u>of a continuous function $P(x)$ of period 2π is always summable and, moreover, uniformly summable, with sum $P(x)$, i.e., the mean value</u>

$$S_n(x) = \frac{1}{n}(s_0(x) + s_1(x) + \cdots + s_{n-1}(x)) = \frac{1}{n}\sum_{\nu=-n}^n (n-|\nu|)a_\nu e^{i\nu x}$$

$$= \sum_{\nu=-n}^n \left(1 - \frac{|\nu|}{n}\right)a_\nu e^{i\nu x}$$

approaches $P(x)$ uniformly for all x as $n \to \infty$.

The proof rests on the transformation of the expression for the mean value $S_n(x)$. Since for a fixed value of x

$$P(x+t) \sim \sum a_n e^{inx}\cdot e^{int},$$

we have first for the nth partial sum

$$s_n(x) = \sum_{\nu=-n}^{n} a_\nu \, e^{i\nu x} = \sum_{\nu=-n}^{n} \underset{t}{M}\{P(x+t)\,e^{-i\nu t}\}$$

$$= \underset{t}{M}\left\{P(x+t)\sum_{\nu=-n}^{n} e^{-i\nu t}\right\} = \underset{t}{M}\{P(x+t)\,D_n(t)\},$$

where the Dirichlet kernel $D_n(t)$ is given by

$$D_n(t) = \sum_{\nu=-n}^{n} e^{-i\nu t} = \frac{e^{i(n+\frac{1}{2})t} - e^{-i(n+\frac{1}{2})t}}{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}} = \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} = \frac{\cos nt - \cos(n+1)t}{2\sin^2\frac{t}{2}}$$

For the mean value $S_n(x)$, we find a corresponding expression

$$S_n(x) = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) a_\nu \, e^{i\nu x} = \underset{t}{M}\{P(x+t)\,K_n(t)\},$$

where $K_n(t)$ is the Féjer kernel given by

$$K_n(t) = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) e^{-i\nu t} = \frac{1}{n} \sum_{\nu=-n}^{n} (n - |\nu|) \, e^{-i\nu t}$$

$$= \frac{1}{n}\left(D_0(t) + D_1(t) + \cdots + D_{n-1}(t)\right)$$

$$= \frac{1}{n}\,\frac{1 - \cos nt}{2\sin^2\frac{t}{2}} = \frac{1}{n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2$$

Of these expressions for $K_n(t)$ the first and (particularly) the last are important. From the first it follows that

$$M\{K_n(t)\} = 1$$

(since the constant term is 1) and from the last that — unlike $D_n(t)$ —

$$K_n(t) \geqq 0$$

for all t. (If t is a multiple of 2π, then $K_n(t) = n$; if however, nt, but not t, is a multiple of 2π, then $K_n(t) = 0$).

Now let $\varepsilon > 0$ be assigned an arbitrary value. We have to show that

$$|S_n(x) - P(x)| < \varepsilon$$
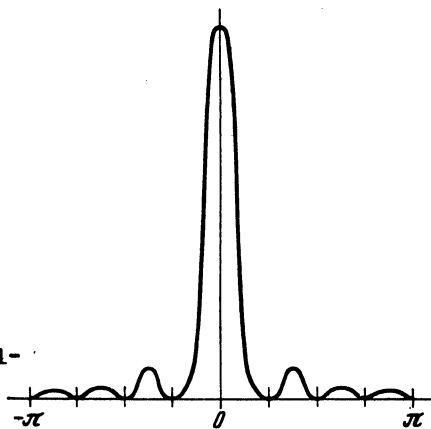
Fig. 2.

for all x, when $n > N = N(\varepsilon)$.   For an arbitrary fixed x

$$S_n(x) - P(x) = \underset{t}{M}\{P(x+t)\,K_n(t)\} - P(x)\,\underset{t}{M}\{K_n(t)\}$$
$$= \underset{t}{M}\{(P(x+t) - P(x))\,K_n(t)\}.$$

Now $P(x)$ is continuous and periodic, hence $P(x)$ is bounded and uniformly continuous, i.e.,

$$|P(x)| \leqq C$$

for all x, and $|P(x_1) - P(x_2)| \leqslant \eta$ for $|x_1 - x_2| \leqq \delta = \delta(\eta)$. We denote by $\omega(\delta)$, the upper limit of $|P(x_1) - P(x_2)|$ for $|x_1 - x_2| \leqq \delta$, and then $\omega(\delta) \to 0$ for $\delta \to 0$.   Since we make use of the specified properties of $K_n(t)$ we obtain now for an arbitrary δ $(< \pi)$

$$|S_n(x) - P(x)| \leqq \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}|P(x+t) - P(x)|\,K_n(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int\limits_{-\delta}^{\delta} + \int\limits_{-\pi}^{-\delta} + \int\limits_{\delta}^{\pi}\right]|P(x+t) - P(x)|\,K_n(t)\,dt$$

$$\leqq \frac{1}{2\pi}\left[\omega(\delta)\int\limits_{-\delta}^{\delta}K_n(t)\,dt + 2(\pi - \delta)\,2C \cdot \frac{1}{n}\,\frac{1}{\sin^2\dfrac{\delta}{2}}\right]$$

$$< \frac{1}{2\pi}\left[\omega(\delta)\int\limits_{-\pi}^{\pi}K_n(t)\,dt + 2\pi\,\frac{1}{n}\,\frac{2C}{\sin^2\dfrac{\delta}{2}}\right]$$

$$= \omega(\delta) + \frac{1}{n}\,\frac{2C}{\sin^2\dfrac{\delta}{2}}\,.$$

We first choose a __fixed__ δ so small that $\omega(\delta) < \varepsilon/2$, and then an N so large that $\dfrac{1}{N}\,\dfrac{2C}{\sin^2\dfrac{\delta}{2}} < \dfrac{\varepsilon}{2}$.   Then for $n > N$ (and all x)

$$|S_n(x) - P(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves the theorem.

33.   The point should be brought out that the proof of Féjer's theorem was given independently of the principal theorems preceding it.   The corresponding theorems in the theory of almost periodic functions have a different inter-

relation. It is, therefore, of interest to note this in-
dependence since, on the other hand, the two principal
theorems can be shown to follow from Féjer's theorem.

Fejer's theorem is seen to contain the <u>uniqueness
theorem</u> on the basis of the fact that the former theorem
gives an <u>algorithm</u> for arriving at the function $P(x)$ from
the Fourier series of $P(x)$

It will now be seen that Féjer's theorem also contains
Parseval's equation.

WEIERSTRASS' THEOREM

34. Fejer's theorem contains as a special case a
classical theorem of Weierstrass which can be considered to
be the true fundamental theorem of the theory of (continu-
ous) periodic functions.

<u>Every continuous function</u> $P(x)$ <u>of period</u> 2π <u>can be</u>
approximated uniformly for all x by trigonometric poly-
nomials $\sum^* c_n e^{inx}$, i.e., for every given $\varepsilon > 0$ a sum $\sum^* c_n e^{inx}$
can be given such that

$$|P(x) - \sum^* c_n e^{inx}| < \varepsilon$$

for all x.

This theorem is contained in Fejer's theorem and it can
be deduced immediately from it if one disregards the special
nature of the trigonometric sum $S_n(x)$. Weierstrass'
theorem states less than Fejer's theorem; for example, it
follows immediately from the latter that in the approxima-

ting sum $\sum^* c_n e^{inx}$ we need use only those harmonics e^{inx} which
actually appear in the Fourier series of $P(x)$, i.e., whose
Fourier coefficients a_n are distinct from zero. The Weier-
strass theorem, however, contains the Parseval's equality,
which, indeed, asserts only the possibility of approximation
of $P(x)$ by trigonometric polynomials in the mean. In order
to have

$$M\{|P(x) - \sum^* c_n e^{inx}|^2\} < \varepsilon$$

it <u>suffices</u> to have

$$|P(x) - \sum^* c_n e^{inx}| < \sqrt{\varepsilon}$$

for all x.

35. For later use it is convenient to have the Weier-
Strass theorem stated in a slightly different form.
We consider the set of <u>finite</u> sums of the form

$$S(x) = \sum{}^* c_n e^{inx}$$

(with arbitrary complex coefficients) and extend this set
by adding to it all functions $f(x)$ which can be <u>approximated
uniformly</u> by such sums $S(x)$. This extended class of func-
tions will be called the <u>closure</u> (abgeschlossene Huelle) of

$\{S(x)\}$ and will be denoted by $H\{S(x)\}$.
   <u>Then this set of functions $H\{S(x)\}$ is identical with</u>
<u>the class of all continuous periodic functions $P(x)$ of period</u>
2π.

   For on one hand, every function belonging to $H\{S(x)\}$ is
continuous and periodic with period 2π since each function
$S(x)$, has these properties and so must every function which
is the limit of a uniformly convergent sequence of func-
tions $S(x)$. On the other hand, every continuous periodic
function $P(x)$ of period 2π belongs to $H\{S(x)\}$, for this
is precisely the Weierstrass theorem.

   36. With this we conclude our study of the theory of
continuous periodic functions. It should be noticed that
one can prove Weierstrass' theorem without recourse to
Fourier series, and that this theorem then provides perhaps
the simplest approach to the fundamental theorems of Fourier
series. Our particular presentation was chosen only because
it provides a basis for the development of the theory of
almost periodic functions. In fact this method is the nat-
ural (and until now the usual) method, since the theory of
almost periodic functions can be discussed simply in terms
of approximation by sums $\sum{}^* c_n e^{i\lambda_n x}$ with arbitrary real ex-
ponents λ_n .

TWO REMARKS

   37. We have hitherto considered only continuous Per-
iodic functions $P(x)$ in anticipation of Weierstrass' theorem
which holds only for continuous functions. The principal
theorems of <u>Fourier theory</u> retain their validity for more

general classes of functions. One such result will be nec-
essary for a later application to the theory of almost per-
iodic functions.

Let $P(x)$ be periodic with period 2π and continuous
everywhere with the exception of the points $0, \pm 2\pi, \pm 4\pi, \ldots$
at each of which $P(x)$ has a saltus or "jump" (i.e., where
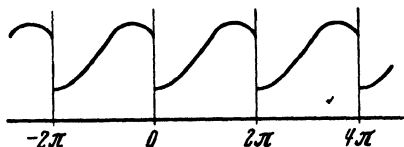
Fig. 3.

right and left hand limits exist but are distinct). Further
let

$$\sum_{-\infty}^{\infty} a_n e^{inx} \quad \text{with} \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} P(x)\, e^{-inx}\, dx$$

be the Fourier series of $P(x)$. We wish to show, that under
these circumstances Parseval's equation

$$(*) \qquad \frac{1}{2\pi}\int_0^{2\pi} |P(x)|^2\, dx = \sum_{-\infty}^{\infty} |a_n|^2$$

remains valid; and, moreover, the following, somewhat
stronger, equation holds:

$$\frac{1}{2\pi}\int_0^{2\pi} P(x+t)\,\overline{P(t)}\, dt = \sum_{-\infty}^{\infty} |a_n|^2 e^{inx}$$

Proof: Exactly as in §24, one can show that the
series $\sum_{-\infty}^{\infty} |a_n|^2$, converges (moreover, with a sum $\leq M\{|P(x)|^2\}$),
and accordingly the series $\sum_{-\infty}^{\infty} |a_n|^2 e^{inx}$, also converges, and in
fact uniformly in x. It also follows, exactly as in §20,
that the folded function

$$Q(x) = \frac{1}{2\pi}\int_0^{2\pi} P(x+t)\,\overline{P(t)}\, dt$$

(which is also periodic of period 2π) has as its Fourier
series exactly the series $\sum_{-\infty}^{\infty} |a_n|^2 e^{inx}$. Now, however, the
function $Q(x)$, (despite the discontinuity of the integrand
$P(x+t)\,\overline{P(t)})$) is an everywhere continuous function of x .

Therefore, we can apply immediately the uniqueness theorem
for continuous functions and get the desired equation

$$Q(x) = \sum_{-\infty}^{\infty} |a_n|^2 e^{inx},$$

which becomes Parseval's equation (*) for $x = 0$.

38. Finally we note that the simple change of variable

$$\frac{2\pi}{T} x = x'$$

makes it possible to extend our results immediately to per-
iodic functions whose period is not 2π, but an arbitrary pos-
itive number T. Let $F(x)$ be such a function, which may
have jumps at the points $mT \, (m = 0, \pm 1, \pm 2, \ldots)$ then the
Fourier series will be of the form

$$F(x) \sim \sum_{-\infty}^{\infty} a_n e^{in\frac{2\pi}{T}x}$$

where

$$a_n = M\left\{F(x) e^{-in\frac{2\pi}{T}x}\right\} = \frac{1}{T}\int_0^T F(x) e^{-in\frac{2\pi}{T}x} \, dx,$$

and likewise Parseval's Equation

$$\frac{1}{T}\int_0^T F(x)|^2 \, dx = \sum_{-\infty}^{\infty} |a_n|^2$$

is valid as well as its generalization

$$G(x) = \frac{1}{T}\int_0^T F(x+t)\,\overline{F(t)} \, dt = \sum_{-\infty}^{\infty} |a_n|^2 e^{in\frac{2\pi}{T}x}.$$

THE THEORY OF

ALMOST PERIODIC FUNCTIONS

THE MAIN PROBLEM OF THE THEORY

39. We consider the set of all finite sums

$$s(x) = \sum_{n=1}^{N} a_n e^{i \lambda_n x}, \qquad -\infty < x < \infty,$$

where the coefficients a_n are arbitrary complex quantities
while the exponents λ_n are arbitrary real quantities. The
set of these functions $s(x)$ will now be "closed" by the addi-
tion of those functions $f(x) = u(x) + iv(x)$ which can be approx-
imated by such sums $s(x)$ <u>uniformly for all</u> x . Thus for any
$\varepsilon > 0$, some $s(x)$ exists satisfying

$$|f(x) - s(x)| \leq \varepsilon \qquad \text{for} \quad -\infty < x < \infty;$$

(These functions $f(x)$ are obviously continuous functions of
x). We call the function class obtained by this extension
the <u>closure</u> of the set $\{s(x)\}$ and we denote it by $H\{s(x)\}$.

Thus the main problem of the theory consists in char-
acterizing the functions $f(x)$ of this class $H\{s(x)\}$ by "struc-
tural properties" bearing no direct relationship to the con-
cepts used in the definition of $H\{s(x)\}$.

Remarks: A fundamental difference exists between this
problem and the corresponding one (solved by Weierstrass)
for the case of harmonic sums $\sum^* a_n e^{inx}$. For in the former
case the exponents λ_n are chosen from the non-enumerable set
of real numbers $-\infty < \lambda < \infty$ and not just from the enumerable
set $\lambda = n (= 0, \pm 1, \pm 2, \dots)$. Later we shall see that the
functions $f(x)$ corresponding to the class $H\{s(x)\}$ help us
overcome this difficulty by permitting us to pick out dis-
tinct enumerable sets of exponents $\lambda_1, \lambda_2, \lambda_3, \dots$ which take the
place of the set of exponents $0, \pm 1, \pm 2, \dots$.

TRANSLATION NUMBERS

40. For a beginning we consider at first a simple
example of a function of the class $H\{s(x)\}$, namely the
function

$$s(x) = e^{i\lambda_1 x} + e^{i\lambda_2 x},$$

where λ_1 and λ_2 denote real numbers distinct from zero.
the two terms $e^{i\lambda_1 x}$ and $e^{i\lambda_2 x}$ individually are each periodic
with the periods respectively of $p_1 = 2\pi/|\lambda_1|$ and $p_2 = 2\pi/|\lambda_2|$
At the same time, of course, all their integral multiples
are also periods of $e^{i\lambda_1 x}$ and $e^{i\lambda_2 x}$. There are two cases:

      1. λ_1/λ_2 is rational, i.e., p_1/p_2 is rational or $n_1 p_1$
$= n_2 p_2$ for two integers n_1 and n_2 distinct from zero. In this
trivial case the two terms $e^{i\lambda_1 x}$ and $e^{i\lambda_2 x}$ have the common
period $P = n_1 p_1 = n_2 p_2$, and their sum $s(x)$ is purely periodic
with this period P.

      2. λ_1/λ_2 is irrational, i.e., p_1/p_2 is irrational.
In this case no multiple of p_1 equals a multiple of p_2 i.e.
the two arithmetic progressions $(0, \pm p_1, \pm 2p_1, \ldots)$ and $(0, \pm p_2, \pm 2p_2, \ldots)$ have the origin as their only common point. In
this case the function $s(x)$ is <u>not</u> periodic: e.g., it has the
value 2 for $x = 0$ and for no other value of x.

On the other hand, as can easily be shown, there exists
for any given $\delta > 0$ a pair of arbitrarily great integers
n_1 and n_2, such that

$$|n_1 p_1 - n_2 p_2| < \delta$$

(This is a special case of a general theorem in Diophantine
Approximation). Let τ be a number which is "near" to $n_1 p_1$
and $n_2 p_2$, (for example, a number between $n_1 p_1$ and $n_2 p_2$)
then τ is "almost" a period of $e^{i\lambda_1 x}$ as well as $e^{i\lambda_2 x}$ and
accordingly "almost" a period of its sum $s(x)$, the differ-
ence $s(x + \tau) - s(x)$ is very small for all x.

    41. From these considerations we shall be led to the
formal definition of a "translation number (Verschiebungs-
zahl)". Let us take an arbitrary function $f(x) = u(x) + iv(x)$
continuous for $-\infty < x < \infty$. The real number τ will then be
called a <u>translation number of</u> $f(x)$ <u>corresponding to</u> ε
(and denoted by $\tau(\varepsilon)$ or $\tau_f(\varepsilon)$) whenever

$$|f(x + \tau) - f(x)| \leqq \varepsilon \quad \text{for} \quad -\infty < x < \infty.$$

<u>Remarks</u>: Obviously a translation number of $f(x)$ corre-
sponding to ε corresponds a fortiori to every quantity $\varepsilon_1 > \varepsilon$
and together with τ, $-\tau$ is also a translation number of
$f(x)$ corresponding to ε. It is further true that $\tau(\varepsilon_1)$
$+ \tau(\varepsilon_2) = \tau(\varepsilon_1 + \varepsilon_2)$ and $\tau(\varepsilon_1) - \tau(\varepsilon_2) = \tau(\varepsilon_1 + \varepsilon_2)$) i.e., the sum
or difference of translation numbers corresponding to ε_1 or

ε_2 , will at the same time be a translation number corresponding to $\varepsilon_1 + \varepsilon_2$.

DEFINITION OF ALMOST PERIODICITY

42. After the special case studied above, it is natural to consider the class of functions $f(x)$ which have the following property: For every $\varepsilon > 0$ there exists a translation number $\tau = \tau(\varepsilon)$ of $f(x)$ and these numbers $\tau(\varepsilon)$ are arbitrarily great. This is not, however, a happy choice. Not only is the class of functions $f(x)$ far from the desired class $H\{s(x)\}$, but it can even be shown, by constructing suitable examples, that this class does not even remain invariant under the ordinary operation of addition, i.e., in case $f(x)$ and $g(x)$ are two functions of this class their sum $f(x) + g(x)$ need not belong to this class.

43. To be able to conclude that $f(x)$ belongs to $H\{s(x)\}$ more must be said about $f(x)$ than merely that it has arbitrarily large translation numbers for each $\varepsilon > 0$. To this end, we introduce the concept of "relative density". A set E of real numbers τ will be called <u>relatively dense</u> (<u>relativ dicht</u>) if there are no arbitrarily large gaps among the numbers τ or, to be exact, if some length L exists such that every interval $\alpha < x < \alpha + L$ of this length contains at least one number τ of the set E.
  <u>Example</u>: The numbers $n.p\ (n = 0, \pm 1, \pm 2, \ldots, p > 0)$ of an arithmetic progression are relatively dense and indeed the simplest such set possible. The set of numbers $\pm n^2\ (n = ($ $1, 2, \ldots)$, on the contrary, is not relatively dense (since $(n + 1)^2 - n^2 = 2n + 1 \to \infty$ as $n \to \infty$). Roughly speaking, a "relatively dense" set can be described as one which is "just as dense" as an arithmetic progression.

44. We now come to the real definition of an "almost periodic" function.
  MAIN DEFINITION. <u>A function continuous for $-\infty < x < \infty$ will be called almost periodic when, given an $\varepsilon > 0$, there exists a relatively dense set of translation numbers of $f(x)$ corresponding to ε . In other words, to every $\varepsilon > 0$, a</u> length $L = L(\varepsilon)$ of some sort exists such that each interval of length $L(\varepsilon)$ contains at least one translation number

$\tau = \tau(\varepsilon)$.

Examples: A continuous purely periodic function of
period p is obviously a special case of an almost periodic
function. In fact here we can take the periods np $(n = 0,$
$\pm 1, \pm 2, \ldots)$ as translation numbers $\tau(\varepsilon)$ for each ε . It
is further seen that if $f(x)$ is almost periodic, the same
is true of $f(x + c)$ (c an arbitrary real constant) of $cf(x)$
(c an arbitrary complex constant) of $\overline{f(x)}$ and also of $|f(x)|$
since $\left| |f(x + \tau)| - |f(x)| \right| \leqq |f(x + \tau) - f(x)|$).

In case $f(x)$ is not purely periodic every usable length
$l(\varepsilon)$ must be arbitrarily large if ε is chosen small enough.
Otherwise it would be possible to use one and the same
length for every ε . For example, the interval $L \leqq x \leqq 2L$
would contain a point of accumulation of arbitrarily fine
translation numbers τ, (i.e., translation numbers corres-
ponding to arbitrarily small values of ε). Such a point
of accumulation, contrary to assumption, would be a perio
of $f(x)$.

The fundamental theorem of the theory of almost per-
iodic functions can be demonstrated only at the end of this
entire chapter. It states, that the class of almost per-
iodic functions is identical with the class $H\{s(x)\}$. Thus
almost periodicity is precisely the desired structural prop-
erty which characterizes the functions $f(x)$ corresponding
to $H\{s(x)\}$

TWO SIMPLE PROPERTIES OF ALMOST PERIODIC FUNCTIONS

45. We wish to develop the theory of almost periodic
functions in direct analogy with the theory of purely per-
iodic functions. Here we must note, however, that many
theorems which are decidedly trivial for purely periodic
functions are no longer trivial for almost periodic func-
tions. This is illustrated by two theorems that we shall
presently prove, according to which every almost periodic
function is bounded and uniformly continuous. The corres-
ponding theorems for purely periodic functions were trivial
because in the investigation of such functions we observed
that we could restrict ourselves to a finite interval. For
almost periodic functions the proof -- a very simple one --
rests on a similar observation which is made possible by the

existence of a length $L = L(\varepsilon)$ as provided in the definition
of almost periodicity.

From the property of the number $L = L(\varepsilon)$ it follows dir-
ectly that if we consider a <u>fixed</u> interval J of length
$L = L(\varepsilon)$ (e.g., the interval $0 < x < L$) we are always able to
find a translation number $\tau = \tau(\varepsilon)$ $(= \tau(\varepsilon, x_0))$ for any x_0 whatso-
ever, such that a translation τ changes x_0 into some point
$x_1 = x_0 + \tau$ of this interval J. (We have only to choose the
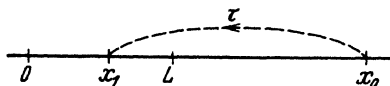
Fig. 4.

translation number so that it belongs to the interval $-x_0$
$< x < -x_0 + L$). Thus even in the case of almost periodic
functions, this consideration makes it possible to restrict
the work to a fixed, finite interval.

46. We now prove the above mentioned theorems.

THEOREM I. <u>An almost periodic function is bounded</u> i.e.,
there exists a constant $C = C(f)$, such that

$$|f(x)| \leq C \quad \text{for} \quad -\infty < x < \infty.$$

<u>Proof</u>: First we specify the length $L = L(1)$ for $\varepsilon = 1$.
In the closed interval $0 \leq x \leq L(1)$ the function $f(x)$ is bound-
ed, say $|f(x)| \leq c$. Then at each point x_0 the inequality
$|f(x_0)| \leq c + 1 = C$ will hold. In fact we could find a trans-
lation number $\tau = \tau(1)$ for an arbitrarily prescribed x_0 such
that $0 < x_0 + \tau < L(1)$ and we thus obtain

$$|f(x_0)| \leq |f(x_0 + \tau)| + |f(x_0) - f(x_0 + \tau)| \leq c + 1.$$

<u>Corollary</u>: If $f(x)$ is almost periodic, so is $(f(x))^2$. In
fact we have

$$|(f(x+\tau))^2 - (f(x))^2| = |f(x+\tau) + f(x)| \cdot |f(x+\tau) - f(x)| \leq 2C|f(x+\tau) - f(x)|,$$

and thus every translation number of $f(x)$ corresponding to
$\varepsilon/2C$ is likewise a translation number of $(f(x))^2$ corres-
ponding to ε.

In particular $|f(x)|^2$ is an almost periodic function.

As a simple application of Theorem I, let us take the
following: If the almost periodic function $f(x) \neq 0$ for all
x, then in order that $1/f(x)$ also be almost periodic it is nec-

essary that the lower limit γ of $|f(x)|$ for $-\infty < x < \infty$ be greater than zero. (This condition is not satisfied automatically). The condition $\gamma > 0$ is, however, sufficient; for according to an inequality valid for all τ , namely,

$$\left|\frac{1}{f(x+\tau)} - \frac{1}{f(x)}\right| = \left|\frac{f(x+\tau) - f(x)}{f(x)\,f(x+\tau)}\right| \leq \frac{1}{\gamma^2}|f(x+\tau) - f(x)|$$

we find that the translation number of $f(x)$ corresponding to $\gamma^2 \varepsilon$ is simultaneously a translation number of $1/f(x)$ corresponding to ε.

THEOREM II: <u>An almost periodic function $f(x)$ is uniformly continuous for $-\infty < x < \infty$</u> (and not merely for any finite interval) i.e., to an arbitrary $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ of such nature that

$$|f(x_1) - f(x_2)| \leq \varepsilon \quad \text{for} \quad |x_1 - x_2| \leq \delta.$$

<u>Proof</u>: ε being given, we first specify the length $L = L\left(\frac{\varepsilon}{3}\right)$ and consider the finite closed interval $-1 \leq x \leq L\left(\frac{\varepsilon}{3}\right) + 1$. We then determine a $\delta(<1)$, so that the inequality $|f(y_1) - f(y_2)| \leq \frac{\varepsilon}{3}$ is valid for every pair of points (y_1, y_2) in the interval $\left(-1, L\left(\frac{\varepsilon}{3}\right) + 1\right)$ provided $|y_1 - y_2| \leq \delta$. Then this δ will have the desired property. In fact let (x_1, x_2) be an arbitrary pair of points with $|x_1 - x_2| \leq \delta$, and let the translation number $\tau = \tau\left(\frac{\varepsilon}{3}\right)$ be chosen so that the point $y_1 = x_1 + \tau$ lies in the interval $0 < x < L\left(\frac{\varepsilon}{3}\right)$ Then the point $y_2 = x_2 + \tau$ obviously lies in the interval $\left(-1, L\left(\frac{\varepsilon}{3}\right) + 1\right)$ and thus we obtain the equation

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(y_1)| + |f(y_1) - f(y_2)| + |f(y_2) - f(x_2)|$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

<u>Remarks</u>: Theorem II is equivalent to the following statement. To every ε there exists a δ, so that <u>every</u> number τ satisfying the condition $|\tau| \leq \delta$ yields a translation number corresponding to ε.

47. From Theorem II we can deduce a corollary which we will use often.

<u>Corollary</u>: For each ε there exists an L <u>and</u> a δ , such

that each interval $\alpha < x < \alpha + L$ of length L contains not just one translation number $\tau(\varepsilon)$, but rather <u>an entire interval of length</u> δ whose points are all translation numbers $\tau(\varepsilon)$.

Proof: Let $L\left(\dfrac{\varepsilon}{2}\right)$ and $\delta\left(\dfrac{\varepsilon}{2}\right)$ respectively, be chosen according to the definition of almost-periodicity and Theorem II. Then the two numbers $L = L\left(\dfrac{\varepsilon}{2}\right) + 2\delta\left(\dfrac{\varepsilon}{2}\right)$ and $\delta = 2\delta\left(\dfrac{\varepsilon}{2}\right)$ fulfill the conditions of this corollary. In fact a translation number $\tau\left(\dfrac{\varepsilon}{2}\right)$, lies in every interval of length $L\left(\dfrac{\varepsilon}{2}\right)$
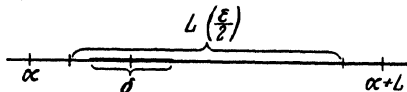
Fig. 5.

and further, each number of the form $\tau\left(\dfrac{\varepsilon}{2}\right) + \beta$ with $|\beta| \leq \delta\left(\dfrac{\varepsilon}{2}\right)$ is a translation number $\tau(\varepsilon)$, since according to Theorem II, the number β is a translation number $\tau\left(\dfrac{\varepsilon}{2}\right)$. So that in every interval of length $L\left(\dfrac{\varepsilon}{2}\right) + 2\delta\left(\dfrac{\varepsilon}{2}\right)$ lies an interval of length $2\delta\left(\dfrac{\varepsilon}{2}\right)$, and all the numbers in the latter interval are translation numbers $\tau(\varepsilon)$.

A more applicable form of the corollary is the somewhat weaker statement: To every ε there exists an L and a δ, so that for every <u>arbitrarily chosen</u> positive quantity $\eta \leq \delta$ each interval $\alpha < x < \alpha + L$ of length L contains a translation number $\tau(\varepsilon)$ which is <u>an integral multiple of</u> η .

THE INVARIANCE OF ALMOST PERIODICITY RELATIVE TO SIMPLE OPERATIONS

48. We now prove a more difficult theorem:

THEOREM III. The sum $f(x) + g(x)$ of two almost periodic functions $f(x)$ and $g(x)$ is an almost periodic function.

Proof: It suffices to show that for each $\varepsilon > 0$ there exists a relatively dense set of <u>common</u> translation numbers $\tau(\varepsilon)$ of the two functions $f(x)$ and $g(x)$. In fact, every such number $\tau = \tau_f(\varepsilon) = \tau_g(\varepsilon)$ is obviously a translation number $\tau_{f+g}(2\varepsilon)$ because

$$\left| (f(x+\tau)+g(x+\tau)) - (f(x)+g(x)) \right| \leq |f(x+\tau) - f(x)| + |g(x+\tau) - g(x)| \leq 2\varepsilon .$$

We use the corollary of §47 in its last form and deter-
mine the numbers L_f, δ_f and L_g, δ_g which correspond to the
given number $\varepsilon/2$ according to this corollary. Let $L_0 =$
Max (L_f, L_g) and $\eta =$ Min (δ_f, δ_g). Then each interval $\alpha < x < \alpha +$
L_0 of length L_0 contains a translation number $\tau_f\left(\frac{\varepsilon}{2}\right)$ as
well as a translation number $\tau_g\left(\frac{\varepsilon}{2}\right)$, which are both multiples
of η (see Fig. 6).
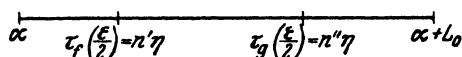
Fig. 6.

We consider all pairs of these translation numbers,
i.e., all pairs $\left(\tau_f\left(\frac{\varepsilon}{2}\right), \tau_g\left(\frac{\varepsilon}{2}\right)\right)$ with $\tau_f = n'\eta$, $\tau_g = n''\eta$ and $|\tau_f -$
$\tau_g| < L_0$. For each pair of this kind, we find

$$\tau_f - \tau_g = (n' - n'')\eta = n\eta,$$

where n is an integer. Since $|n\eta| < L_0$ there can be only
finitely many distinct values of $n\eta$. Let these be $n_1\eta$,
$n_2\eta, \ldots, n_Q\eta$ and let them be "represented" by the point
pairs $(\tau_f^{(1)}, \tau_g^{(1)}), (\tau_f^{(2)}, \tau_g^{(2)}), \ldots, (\tau_f^{(Q)}, \tau_g^{(Q)})$
which are arbitrary, but, once chosen, are kept fixed.
Further, let us write

$$\underset{q=1, 2, \ldots, Q}{\text{Max}} |\tau_f^{(q)}| = l$$

We shall show that each interval of length $L_0 + 2l$ contains
at least one common translation number $\tau = \tau_f(\varepsilon) = \tau_g(\varepsilon)$.
Let $\alpha < x < \alpha + L_0 + 2l$ be an arbitrary interval of
length $L_0 + 2l$. In the interval $\alpha + l < x < \alpha + L_0 + l$ of
length L_0
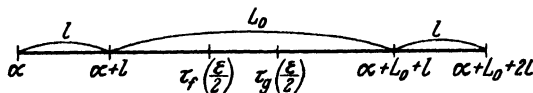
Fig. 7.

we choose (see Fig. 7) two translation numbers $\tau_f\left(\frac{\varepsilon}{2}\right) = n'\eta$
and $\tau_g\left(\frac{\varepsilon}{2}\right) = n''\eta$. Let $n_q\eta$ be that one of the above multiples
of η, for which $\tau_f - \tau_g = n_q\eta$, and let $\tau_f^{(q)}, \tau_g^{(q)}$ be the pre-
viously chosen fixed "representation pair" ("Repraesentant-
enpaar") for this $n_q\eta$. Then $\tau_f - \tau_g = \tau_f^{(q)} - \tau_g^{(q)}$, or $\tau_f - \tau_f^{(q)} =$

$\tau_g - \tau_g^{(q)}$, and this last number
$$\tau = \tau_f - \tau_f^{(q)} = \tau_g - \tau_g^{(q)}$$

is what we have been looking for. In fact it is first of
all a translation number of $f(x)$ as well as of $g(x)$, corres-
ponding to ε , since it is the difference of two transla-
tion numbers corresponding to $\varepsilon/2$. Secondly, it lies in
the interval $(\alpha, \alpha + L_0 + 2l)$, since τ_f belongs to the interval
$(\alpha + l, \alpha + L_0 + l)$ and $|\tau_f^{(q)}| \leqq l$.

    <u>Corollary</u>: The sum $f(x) = \sum_1^N p_n(x)$ of a finite number of

continuous periodic functions $p_n(x)$ with arbitrary periods is
almost periodic. In particular, <u>every trigonometric poly-
nomial,</u>
$$s(x) = \sum_1^N a_n e^{i\lambda_n x}$$

<u>is an almost periodic function.</u>
    (Since $-g(x)$ is almost periodic whenever $g(x)$ is, it
follows from Theorem III that the difference $f(x) - g(x)$ is an
almost periodic function).
    THEOREM IV. The product $f(x)g(x)$ of two almost periodic
functions $f(x)$ and $g(x)$ is also almost periodic.
    <u>Proof</u>: It has been shown before, that the <u>square</u> of
an almost periodic function is also almost periodic. The
theorem in question, therefore, follows from Theorem III
because of the identity
$$f(x)\,g(x) = \tfrac{1}{4}\{(f(x) + g(x))^2 - (f(x) - g(x))^2\}.$$

    49. We have, finally, the following rather simple
theorem.
    THEOREM V. <u>The limit function</u> $f(x)$ <u>of a sequence of
almost periodic functions</u> $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ <u>uniformly
convergent for</u> $-\infty < x < \infty$ <u>is also almost periodic.</u>
    <u>Proof</u>: We notice first that $f(x)$ is continuous. Now
let $\varepsilon > 0$ be arbitrarily given, and let $N = N(\varepsilon)$ be chosen so
large that
$$|f(x) - f_N(x)| \leqq \frac{\varepsilon}{3} \text{ for } -\infty < x < \infty.$$

Then every translation number $\tau = \tau_{f_N}\!\left(\dfrac{\varepsilon}{3}\right)$ is surely a trans-
lation number $\tau_f(\varepsilon)$. For we see that

$$|f(x + \tau) - f(x)| \leqq |f(x + \tau) - f_N(x + \tau)| + |f_N(x + \tau) - f_N(x)| +$$

$$|f_N(x) - f(x)| \leqq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since the translation numbers $\tau_{f_N}\left(\frac{\varepsilon}{3}\right)$ are relatively dense, the same is true of the translation numbers $\tau_f(\varepsilon)$.

   Corollary. Every function $f(x)$, which can be approximated uniformly for all x by finite sums $s(x) = \sum_{1}^{N} a_n e^{i\lambda_n x}$ is almost periodic. In other words: <u>Every function $f(x)$ of the class $H\{s(x)\}$ is an almost periodic function</u>. Thus we have proved one part of our fundamental theorem. It is, however, the remaining part that offers the real difficulty.

THE MEAN VALUE THEOREM

   50.   Our next problem will be to lay the foundation for a theory of Fourier series of almost periodic functions. The key to this problem is Theorem IV together with the following theorem.

   MEAN VALUE THEOREM.   <u>For every almost periodic function there exists a mean value</u>

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\, dx = M\{f(x)\},$$

<u>i.e., the expression $\frac{1}{T} \int_0^T f(x)\, dx$ approaches a definite finite</u> limit as $T \to \infty$. We shall denote this by $M\{f(x)\}$.

   Remarks: In case the function, $f(x)$, is purely periodic for example with period p, the theorem is trivial, and the mean value is in agreement with the earlier mean value

$$M\{f(x)\} = \frac{1}{p} \int_0^p f(x)\, dx .$$

In fact, we find that if we set $T = mp + r$ $(0 \leqq r < p)$ then

$$\int_0^T f(x)\, dx = m \int_0^p f(x)\, dx + \int_{mp}^{mp+r} f(x)\, dx ,$$

from which the desired relation

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\, dx = \frac{1}{p} \int_0^p f(x)\, dx$$

will follow, because the limit $\frac{m}{T} \to \frac{1}{p}$ as $T \to \infty$, (considering the fact that $f(x)$ is bounded).

Proof: We shall apply the general convergence principle. Accordingly, we let $\varepsilon > 0$ be given arbitrarily. We have to show the existence of some number $T_0 = T_0(\varepsilon)$ such that the inequality

$$\left| \frac{1}{T_1} \int_0^{T_1} f(x)\, dx - \frac{1}{T_2} \int_0^{T_2} f(x)\, dx \right| < 2\varepsilon$$

holds for $T_1 > T_0, T_2 > T_0$. The proof of this will require several steps.

1. We denote by l_0 the length $L\left(\frac{\varepsilon}{2}\right)$ corresponding to a given $\varepsilon > 0$. For a given fixed number $T > 0$ and a given variable number α we wish to approximate the difference

$$\frac{1}{T} \int_0^T f(x)\, dx - \frac{1}{T} \int_\alpha^{\alpha + T} f(x)\, dx$$

This is done as follows on the basis of the almost-periodicity of $f(x)$:

In the interval $(\alpha, \alpha + l_0)$ we choose a translation number $\tau = \tau_f\left(\frac{\varepsilon}{2}\right)$ and obtain the difference under consideration in the form

$$\left\{ \frac{1}{T} \int_0^T f(x)\, dx - \frac{1}{T} \int_\tau^{\tau + T} f(x)\, dx \right\} - \frac{1}{T} \int_\alpha^\tau f(x)\, dx + \frac{1}{T} \int_{\alpha + T}^{\tau + T} f(x)\, dx.$$

Now the first term $\{...\}$ is numerically $\leq \frac{\varepsilon}{2}$, because $|f(x + \tau) - f(x)| \leq \frac{\varepsilon}{2}$, and each of the two remaining terms is numerically $\leq \frac{1}{T} l_0 \Gamma$, where we denote by Γ the (finite) upper limit of $|f(x)|$ in $-\infty < x < \infty$. Thus we obtain:

(1) $$\left| \frac{1}{T} \int_0^T f(x)\, dx - \frac{1}{T} \int_\alpha^{\alpha + T} f(x)\, dx \right| \leq \frac{\varepsilon}{2} + \frac{2 l_0 \Gamma}{T}.$$

2. Our next problem will be to estimate the difference

$$\frac{1}{T}\int_0^T f(x)\,dx - \frac{1}{nT}\int_0^{nT} f(x)\,dx$$

where $T > 0$ is given, and the positive integer n is arbitrary. This difference is simply the arithmetic mean of the n differences

$$\frac{1}{T}\int_0^T f(x)\,dx - \frac{1}{T}\int_{(\nu-1)T}^{\nu T} f(x)\,dx, \qquad \nu = 1, 2, \ldots, n;$$

each of which is according to (1), numerically $\leq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T}$.
Thus the same inequality is true of the arithmetic mean of these differences. Thus we have

$$(2) \qquad \left|\frac{1}{T}\int_0^T f(x)\,dx - \frac{1}{nT}\int_0^{nT} f(x)\,dx\right| \leq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T}.$$

3. Let us further prescribe $T_1 > 0$ and $T_2 > 0$ whose ratio T_1/T_2 we first take to be <u>rational</u>. Then we find two positive integers n_1 and n_2. such that $n_1 T_1 = n_2 T_2$. Now by (2) , however,

$$\left|\frac{1}{T_1}\int_0^{T_1} f(x)\,dx - \frac{1}{n_1 T_1}\int_0^{n_1 T_1} f(x)\,dx\right| \leq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T_1}$$

and

$$\left|\frac{1}{T_2}\int_0^{T_2} f(x)\,dx - \frac{1}{n_2 T_2}\int_0^{n_2 T_2} f(x)\,dx\right| \leq \frac{\varepsilon}{2} + \frac{2l_0\Gamma}{T_2}.$$

Since $n_1 T_1 = n_2 T_2$, it follows that

$$(3) \qquad \left|\frac{1}{T_1}\int_0^{T_1} f(x)\,dx - \frac{1}{T_2}\int_0^{T_2} f(x)\,dx\right| \leq \varepsilon + 2l_0\Gamma\left(\frac{1}{T_1} + \frac{1}{T_2}\right).$$

Now, from considerations of continuity the relation (3) (proved at first only for rational values of the ratio T_1/T_2) must hold for <u>arbitrary (positive)</u> values of T_1 and T_2 .
After this preparation, we can now proceed immediately with the proof of the mean value theorem. Specifically, we choose $T_0 = \dfrac{4 l_0 \Gamma}{\varepsilon} = \dfrac{4\Gamma L\left(\frac{\varepsilon}{2}\right)}{\varepsilon}$, so that when $T_1 > T_0, T_2 > T_0$,
we obviously have

$$\varepsilon + 2 l_0 \Gamma\left(\frac{1}{T_1} + \frac{1}{T_2}\right) < 2\varepsilon.$$

Thus from (3), it follows that

$$\left|\frac{1}{T_1}\int_0^{T_1} f(x)\,dx - \frac{1}{T_2}\int_0^{T_2} f(x)\,dx\right| < 2\varepsilon.$$

Remarks: For later applications it will be important to know exactly how large T should be chosen in order to make sure that the "finite" mean value $\dfrac{1}{T}\displaystyle\int_0^T f(x)\,dx$ actually represents the mean value $M\{f(x)\}$ to within a given degree of accuracy. Such a result can be obtained from the preceding proof. Now that the existence of the mean value $M\{f(x)\}$ is assured, we need only to perform the limiting process $n \to \infty$ in equation (2) . We obtain

$$\left|\frac{1}{T}\int_0^T f(x)\,dx - M\{f(x)\}\right| \leqq \frac{\varepsilon}{2} + \frac{2 l_0 \Gamma}{T}$$

and from this follows the desired result

(4) $\qquad \left|\dfrac{1}{T}\displaystyle\int_0^T f(x)\,dx - M\{f(x)\}\right| < \varepsilon \ \ \text{for } T > T_0 = \dfrac{4\,\Gamma L\left(\dfrac{\varepsilon}{2}\right)}{\varepsilon}.$

In conclusion, it should now be mentioned that the mean values for almost periodic functions can be manipulated by the usual rules. For example $M\{c f(x)\} = c M\{f(x)\}$ (c an arbitrary complex constant) and

$$M\{(f_1(x) + f_2(x))\} = M\{f_1(x)\} + M\{f_2(x)\}.$$

It is important for applications to notice that if $f_1(x), f_2(x),$ $\ldots, f_n(x), \ldots$ denotes a sequence of almost periodic functions, which converges uniformly to the limit function $f(x)$ (which is eo ipso almost periodic) the limit equation is valid, or

$$M\{f(x)\} = \lim_{n \to \infty} M\{f_n(x)\}.$$

This follows immediately from the fact that for an arbitrary n, we have

$$M\{f(x)\} - M\{f_n(x)\} = M\{(f(x) - f_n(x))\}$$

and accordingly

$$|M\{f(x)\} - M\{f_n(x)\}| \leqq M\{|f(x) - f_n(x)|\} \leqq \underset{-\infty < x < \infty}{\text{upper bound}} |f(x) - f_n(x)|,$$

where the last quantity converges to zero as $n \to \infty$.

51. As the first application of inequality (4), we consider the set of all functions $f(x + a)$, where a denotes an arbitrary real constant. Each one of these functions is almost periodic and therefore possesses a mean value

$$(5) \qquad \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x + a)\, dx = M\{f(x + a)\}.$$

We wish to show that the limit <u>holds uniformly with respect to a</u>, or for a given $\varepsilon > 0$ a corresponding $T_0 = T_0(\varepsilon)$ can be chosen independently of a, so that the following inequality holds <u>simultaneously for all values of a</u>.

$$\left| \frac{1}{T} \int_0^T f(x + a)\, dx - M\{f(x + a)\} \right| < \varepsilon \quad \text{for } T > T_0.$$

This, however, is an immediate consequence of the above result, for indeed, the number $T_0 = \dfrac{4\,\Gamma L\left(\dfrac{\varepsilon}{2}\right)}{\varepsilon}$ is the same for all functions $f(x + a)$, since the upper limit of $|f(x + a)|$ equals Γ and $f(x + a)$ has precisely the same translation numbers as $f(x)$.

52. The result of § 51 can also be expressed in another form. To this end we first notice that for constant values of a, obviously

$$M\{f(x + a)\} = M\{f(x)\}$$

and indeed,

$$\frac{1}{T} \int_0^T f(x + a)\, dx = \frac{1}{T} \int_a^{a+T} f(x)\, dx = \frac{1}{T} \int_a^0 f(x)\, dx + \frac{1}{T} \int_0^T f(x)\, dx + \frac{1}{T} \int_T^{a+T} f(x)\, dx,$$

which approaches $M\{f(x)\}$ as $T \to \infty$, since the two other

terms converge to zero (each being bounded $\leqq \frac{1}{T}|a|\Gamma$). The above equation, uniform with respect to a , can now be written as

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x + a)\, dx = M\{f(x)\}$$

or

$$\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} f(x)\, dx = M\{f(x)\}$$

We thus obtained the following theorem.

STRENGTHENED MEAN VALUE THEOREM: <u>The following limit is approached uniformly with respect to</u> a .

$$\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} f(x)\, dx = M\{f(x)\}.$$

This strengthened mean value theorem is a theorem that will be applied frequently below. It teaches us that the mean value of $f(x)$ over an arbitrary finite interval comes "close" to the actual mean value $M\{f(x)\}$, if the interval is chosen sufficiently large, independently of its position. As a trivial application, it might be mentioned that

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T}^0 f(x)\, dx = M\{f(x)\} \quad \text{and} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T f(x)\, dx = M\{f(x)\}.$$

53. In order not to have to break up our train of thought later, we prove here another application of the inequality (4).

Given $f(x)$, an arbitrary almost periodic function, we consider the set of all functions $F_x(t) = f(x + t)\overline{f(t)}$ with x an arbitrary real constant. According to Theorem IV each one of the functions $f(x + t)\overline{f(t)}$ is an almost periodic function of t and accordingly, each possesses a mean value

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x + t)\,\overline{f(t)}\, dt = M_t\{f(x + t)\,\overline{f(t)}\}.$$

We wish to show that this limit <u>holds uniformly with respect to</u> x , or that for a given $\varepsilon > 0$ a corresponding $T_0 = T_0(\varepsilon)$ of some sort can be chosen so that the inequality

$$(6) \qquad \left| \frac{1}{T} \int_0^T f(x + t)\,\overline{f(t)}\, dt - M_t\{f(x + t)\,\overline{f(t)}\} \right| < \varepsilon \text{ for } T > T_0$$

holds simultaneously for all x .

The proof is analogous to the preceding. According to the inequality (4) , the inequality

$$\left|\frac{1}{T}\int_0^T f(x+t)\,\overline{f(t)}\,dt - \underset{t}{M}\{f(x+t)\,\overline{f(t)}\}\right| < \varepsilon \text{ for } T > T_{0,x} = \frac{4\,\Gamma_x L_x\left(\frac{\varepsilon}{2}\right)}{\varepsilon},$$

holds when Γ_x and $L_x\left(\frac{\varepsilon}{2}\right)$ are the appropriate numbers corresponding to the function $F_x(t) = f(x+t)\overline{f(t)}$. Now since it is always true that

$$|f(x+t)\,\overline{f(t)}\,| \leqq \Gamma^2;$$

it follows that for all x , $\Gamma_x \leqq \Gamma^2$. Further for an arbitrary number τ

$$F_x(t+\tau) - F_x(t)$$
$$= f(x+t+\tau)\,\overline{f(t+\tau)} - f(x+t)\,\overline{f(t)}$$
$$= [f(x+t+\tau) - f(x+t)]\cdot\overline{f(t+\tau)} + f(x+t)\cdot\left[\overline{f(t+\tau)} - \overline{f(t)}\right]$$

and thus

$$|F_x(t+\tau) - F_x(t)|$$
$$\leqq |f(x+t+\tau) - f(x+t)|\cdot\Gamma + \Gamma\cdot|\overline{f(t+\tau)} - \overline{f(t)}|$$
$$= \Gamma\{|f(x+t+\tau) - f(x+t)| + |f(t+\tau) - f(t)|\},$$

so that the translation number τ of $f(x)$ corresponding to $\varepsilon/4\Gamma$ is a translation number of $F_x(t)$ corresponding to $\varepsilon/2$. Thus the length $L\left(\frac{\varepsilon}{4\Gamma}\right)$ corresponding to $f(x)$ can be used as the length $L_x\left(\frac{\varepsilon}{2}\right)$. But now we have shown that the inequality (6) holds for

$$T > T_0 = \frac{4\,\Gamma^2 L\left(\frac{\varepsilon}{4\Gamma}\right)}{\varepsilon}$$

where $T_0 = T_0(\varepsilon)$ is independent of x .

54. In conclusion, for later application, we wish to investigate the mean value

$$g(x) = \underset{t}{M}\{f(x+t)\,\overline{f(t)}\}$$

as a function of x . We wish to show that $g(x)$ is an __almost
periodic function.__ To this end we first show that $g(x)$ is
__continuous.__ This follows immediately from the uniform con-
tinuity of $f(x)$. In fact, from

$$|f(x_1) - f(x_2)| \le \varepsilon \qquad \text{for} \qquad |x_1 - x_2| \le \delta = \delta(\varepsilon),$$

it follows that for $|h| \le \delta$

$$|g(x + h) - g(x)| = \left| M_t \{ [f(x + h + t) - f(x + t)] \cdot \overline{f(t)} \} \right| \le \varepsilon \Gamma.$$

That $g(x)$ is __almost periodic__ follows from the fact that
every translation number τ of $f(x)$ corresponding to ε/Γ
is a translation number of $g(x)$ corresponding to ε , since

$$|g(x + \tau) - g(x)| = \left| M_t \{ [f(x + \tau + t) - f(x + t)] \cdot \overline{f(t)} \} \right| \le \frac{\varepsilon}{\Gamma} \cdot \Gamma = \varepsilon$$

In just the same way a simpler result follows, namely, that
the function

$$g_T(x) = \frac{1}{T} \int_0^T f(x + t) \overline{f(t)} \, dt$$

is an almost periodic function for each $T > 0$.

　　The __mean value__ $M\{g(x)\}$ of $g(x)$ can be calculated in the
following manner: Since the function $g_T(x)$ converges uni-
formly to $g(x)$ as $T \to \infty$, according to the result of § 53,
then by an earlier remark,

$$M\{g(x)\} = \lim_{T \to \infty} M\{g_T(x)\}.$$

However, for each fixed T ,

$$\cdot \ \ M\{g_T(x)\} = \lim_{X \to \infty} \frac{1}{X} \int_0^X g_T(x) \, dx = \lim_{X \to \infty} \frac{1}{X} \int_0^X \left(\frac{1}{T} \int_0^T f(x + t) \overline{f(t)} \, dt \right) dx,$$

and by change of the order of integration

$$= \lim_{X \to \infty} \frac{1}{T} \int_0^T \left(\frac{1}{X} \int_0^X f(x + t) \overline{f(t)} \, dx \right) dt = \lim_{X \to \infty} \frac{1}{T} \int_0^T \left(\frac{1}{X} \int_0^X f(x + t) \, dx \right) \overline{f(t)} \, dt.$$

In the last integral, however, the quantity $\frac{1}{X} \int_0^X f(x + t) \, dx$

approaches the mean value $\underset{x}{M}\{f(x + t)\} = \underset{x}{M}\{f(x)\}$ as $X \to \infty$ and,

in fact, it does so uniformly in the interval $-\infty < t < \infty$,
and all the more so in the finite interval of integration
$0 \le t \le T$.　　It follows from this, however (since the pre-

sence of a bounded factor $\overline{f(t)}$ does not disturb the uniformity), that we can let $X \to \infty$ under the integral sign

$\frac{1}{T} \int\limits_{0}^{T} \cdots dt$ so that

$$M\{g_T(x)\} = \frac{1}{T}\int\limits_{0}^{T}\underset{x}{M}\{f(x)\}\,\overline{f(t)}\,dt = \underset{x}{M}\{f(x)\}\frac{1}{T}\int\limits_{0}^{T}\overline{f(t)}\,dt.$$

Finally from this it follows that

$$M\{g(x)\} = \lim_{T \to \infty} M\{g_T(x)\} = \underset{x}{M}\{f(x)\}\underset{t}{M}\{\overline{f(t)}\},$$

And thus (*) $$M\{g(x)\} = |M\{f(x)\}|^2.$$

This relation will be applied later.

Remarks: Formally we could have obtained immediately

$$M\{g(x)\} = \underset{x}{M}\{\underset{t}{M}\{f(x+t)\,\overline{f(t)}\}\} = \underset{t}{M}\{\underset{x}{M}\{f(x+t)\,\overline{f(t)}\}\}$$

$$= \underset{t}{M}\{\overline{f(t)}\,\underset{x}{M}\{f(x+t)\}\} = \underset{t}{M}\{\overline{f(t)}\,\underset{x}{M}\{f(x)\}\} = \underset{x}{M}\{f(x)\}\cdot\underset{t}{M}\{\overline{f(t)}\}.$$

We have interchanged the two mean value operators $\underset{x}{M}$ and $\underset{t}{M}$,

which is in general not permissible since they require taking the mean over an infinite interval. The above proof shows us now, that in this particular case, it actually is true that

$$\underset{x}{M}\{\underset{t}{M}\{f(x+t)\,\overline{f(t)}\}\} = \underset{t}{M}\{\underset{x}{M}\{f(x+t)\,\overline{f(t)}\}\}.$$

We have derived this relation from the trivial interchange

$$\frac{1}{X}\int\limits_{0}^{X}\left(\frac{1}{T}\int\limits_{0}^{T}f(x+t)\,\overline{f(t)}\,dt\right)dx = \frac{1}{T}\int\limits_{0}^{T}\left(\frac{1}{X}\int\limits_{0}^{X}f(x+t)\,\overline{f(t)}\,dx\right)dt$$

by a process based on the above uniformity theorems.

<div align="center">

THE CONCEPT OF THE FOURIER SERIES
OF AN ALMOST PERIODIC FUNCTION.
SETTING UP OF THE PARSEVAL'S EQUATION

</div>

55. We consider the system of <u>all pure vibrations</u> $e^{i\lambda x}$ where λ is an arbitrary real number. Every function $e^{i\lambda x}$, of this non-enumerable system is periodic but the periods are distinct. In contrast to the case of harmonic vibrations e^{inx} $(n = 0, \pm 1, \pm 2, \ldots)$ where we could restrict ourselves to

the finite interval $0 \leqq x \leqq 2\pi$ (since the functions e^{inx}
have the common period 2π), we must consider the <u>infinite
interval</u> $-\infty < x < \infty$. In this interval $-\infty < x < \infty$, our
non-enumerable system $\{e^{i\lambda x}\}$ is a normalized orthogonal sys-
tem in the sense that

$$M\{e^{i\lambda_1 x} e^{-i\lambda_2 x}\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{i\lambda_1 x} e^{-i\lambda_2 x} dx = \begin{cases} 0 & \text{for} & \lambda_1 \neq \lambda_2 \\ 1 & \text{for} & \lambda_1 = \lambda_2 . \end{cases}$$

This is an immediate consequence of

$$\int_0^T e^{i\lambda_1 x} e^{-i\lambda_2 x} dx = \int_0^T e^{i(\lambda_1 - \lambda_2)x} dx = \begin{cases} \dfrac{e^{i(\lambda_1 - \lambda_2)T} - 1}{i(\lambda_1 - \lambda_2)} & \text{for} & \lambda_1 \neq \lambda_2 \\ T & \text{for} & \lambda_1 = \lambda_2 . \end{cases}$$

56. Let $f(x)$ now be an arbitrary almost periodic func-
tion; then for each real λ , the function $g(x) = f(x)e^{-i\lambda x}$
is also an almost periodic function, being the product of
the almost periodic function $f(x)$ and the purely periodic
function $e^{-i\lambda x}$. Consequently its mean value exists. Thus

$$M\{f(x)e^{-i\lambda x}\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x) e^{-i\lambda x} dx .$$

We denote this number $M\{f(x)e^{-i\lambda x}\}$ by $a(\lambda)$, so that we have
defined a function $a(\lambda)$, corresponding to the almost periodic
function $f(x)$ for $-\infty < \lambda < \infty$.
    In our theory the following theorem (to be proved later)
is of fundamental importance:
    <u>The function</u> $a(\lambda) = M\{f(x)e^{-i\lambda x}\}$ <u>is zero for all values of</u>
λ <u>with the exception of an at most enumerable set of num-</u>
bers λ.
    It is this fact which will permit us to carry the
theory of Fourier series over into the field of almost
periodic functions.

57. As in § 14 we first derive the following formula.
    Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be arbitrary ,distinct, real numbers and
let $c_1, c_2, \ldots, c_N$ be arbitrary complex numbers. Then

$$(1) \quad M\left\{|f(x) - \sum_1^N c_n e^{i\lambda_n x}|^2\right\} = M\{|f(x)|^2\} - \sum_1^N |a(\lambda_n)|^2 + \sum_1^N |c_n - a(\lambda_n)|^2.$$

<u>Proof:</u> We notice, first, that the mean values encount-
ered in this formula have meaning, for the two functions $f(x)$

and $f(x) - \sum_1^N c_n e^{i\lambda_n x}$ and likewise the squares of their absolute values) are almost periodic. The formula follows from the following calculation:

$$M\left\{\left|f(x) - \sum_1^N c_n e^{i\lambda_n x}\right|^2\right\} = M\left\{\left(f(x) - \sum_1^N c_n e^{i\lambda_n x}\right)\left(\overline{f(x)} - \sum_1^N \overline{c_n} e^{-i\lambda_n x}\right)\right\}$$

$$= M\left\{f(x)\,\overline{f(x)}\right\} - \sum_1^N \overline{c_n}\, M\left\{f(x)\, e^{-i\lambda_n x}\right\} - \sum_1^N c_n\, M\left\{\overline{f(x)}\, e^{i\lambda_n x}\right\}$$

$$+ \sum_{n_1=1}^N \sum_{n_2=1}^N c_{n_1} \overline{c_{n_2}}\, M\left\{e^{i\lambda_{n_1} x} e^{-i\lambda_{n_2} x}\right\}$$

$$= M\{|f(x)|^2\} - \sum_1^N \overline{c_n}\, a(\lambda_n) - \sum_1^N c_n\, \overline{a(\lambda_n)} + \sum_{n=1}^N c_n \overline{c_n}$$

$$= M\{|f(x)|^2\} + \sum_1^N (c_n - a(\lambda_n))(\overline{c_n} - \overline{a(\lambda_n)}) - \sum_1^N a(\lambda_n)\,\overline{a(\lambda_n)}$$

$$= M\{|f(x)|^2\} - \sum_1^N |a(\lambda_n)|^2 + \sum_1^N |c_n - a(\lambda_n)|^2.$$

    58. If in formula (1) the numbers $a(\lambda_n)$ are chosen for the constants c_n , then there follows the formula:

$$(2) \qquad M\left\{\left|f(x) - \sum_1^N a(\lambda_n)\, e^{i\lambda_n x}\right|^2\right\} = M\{|f(x)|^2\} - \sum_1^N |a(\lambda_n)|^2.$$

    Since the left hand member of formula (2) is obviously a real, non-negative number (being the mean value of a real, non-negative function), this formula yields the following inequality immediately:

$$(3) \qquad \sum_1^N |a(\lambda_n)|^2 \leqq M\{|f(x)|^2\}.$$

    Let us   denote $M\{|f(x)|^2\}$ by C . Then for every positive d there can be only a finite number of values of λ for which $|a(\lambda)| > d$, (certainly fewer than C/d^2). We consider first the set of numbers λ, for which $|a(\lambda)| > 1$, and denote these numbers for instance by

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{n_1};$$

Then we consider the set of those λ, for which $1 \geqq |a(\lambda)| > \tfrac{1}{2}$, and we denote these by

$$\lambda_{n_1+1},\ \lambda_{n_1+2},\ \ldots,\ \lambda_{n_2};$$

After this, we consider the set of these numbers λ , for which $\tfrac{1}{2} \geqq |a(\lambda)| > \tfrac{1}{3}$ etc. In this way we obtain <u>the set of</u>

all λ for which $|a(\lambda)| > 0$, or for which $a(\lambda) \neq 0$, in the form of an ordered sequence, and thus we have shown that $a(\lambda)$ is zero for all λ with the exception of an at most enumerable set of values of λ. The exceptional values of λ (in any order) we shall denote by

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

and we shall call them the <u>Fourier exponents of</u> $f(x)$. The corresponding mean values

$$a(\Lambda_1) = A_1, \quad a(\Lambda_2) = A_2, \quad a(\Lambda_3) = A_3, \quad \ldots$$

we shall call the <u>Fourier coefficients</u> of $f(x)$. We shall further associate with $f(x)$, the finite or infinite series $\sum A_n e^{i\Lambda_n x}$, which we shall call its <u>Fourier series</u> and we shall indicate the formal correspondence by

$$f(x) \sim \sum A_n e^{i\Lambda_n x}.$$

Sometimes it is convenient to use the notation $f(x) \sim \sum a(\Lambda_n) e^{i\Lambda_n x}$, or even $f(x) \sim \sum a(\lambda) e^{i\lambda x}$, without expressly naming the exponents Λ_n . In the last notation, the series represents a <u>formal</u> sum over the non-enumerable set of all numbers $-\infty < \lambda < \infty$.

59. In the special case where $f(x)$ is <u>purely periodic</u> with period 2π, the new definition of Fourier series reduces to the previous one (except for such trivial considerations as the ordering of the terms or the permissibility of terms with coefficient zero). This is far from trivial; though, because of the periodicity of $f(x)e^{-inx}$, it is clear that if λ is an integer n

$$a(\lambda) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x) e^{-inx} dx = \lim_{m \to \infty} \frac{1}{m \, 2\pi} \int_0^{m\,2\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx,$$

we must show more, namely, that when λ <u>is not an integer</u>

$$a(\lambda) = M\{f(x) e^{-i\lambda x}\} = 0 .$$

This follows most simply from the Weierstrass theorem, which says that for a given $\varepsilon > 0$ the function $f(x)$ can be written in the form

$$f(x) = \sum{}^* b_n e^{inx} + R(x)$$

where $|R(x)| \leqq \varepsilon$ for all x. From this, finally, it follows that

$$M\{f(x)\,e^{-i\lambda x}\} = \sum{}^* b_n M\{e^{i(n-\lambda)x}\} + M\{R(x)\,e^{-i\lambda x}\} = M\{R(x)\,e^{-i\lambda x}\},$$

so that

$$\big|\,a(\lambda)\,\big| = \big|\,M\{R(x)\,e^{-i\lambda x}\}\,\big| \leqq \varepsilon\;.$$

Since this must hold for every $\varepsilon > 0$, $\underline{a(\lambda) = 0 \text{ must vanish.}}$

It should be noticed that the last proof can also be given by a rather primitive method without resorting to Weierstrass's theorem. If λ is not an integer, indeed, $e^{-i\lambda 2\pi} \neq 1$. Thus

$$\frac{1}{m\,2\pi}\int\limits_{0}^{m\,2\pi} f(x)\,e^{-i\lambda x}\,dx = \frac{1}{m\,2\pi}\bigg\{\int\limits_{0}^{2\pi} + \int\limits_{2\pi}^{4\pi} + \cdots + \int\limits_{(m-1)2\pi}^{m\,2\pi} f(x)\,e^{-i\lambda x}\,dx\bigg\}$$

$$= \frac{1}{m}\big\{1 + e^{-i\lambda 2\pi} + \cdots + e^{-i\lambda(m-1)\,2\pi}\big\}\frac{1}{2\pi}\int\limits_{0}^{2\pi} f(x)\,e^{-i\lambda x}\,dx$$

$$= \frac{1}{m}\frac{1 - e^{-i\lambda m\,2\pi}}{1 - e^{-i\lambda 2\pi}}\frac{1}{2\pi}\int\limits_{0}^{2\pi} f(x)\,e^{-i\lambda x}\,dx$$

and consequently,

$$a(\lambda) = \lim_{m\to\infty}\frac{1}{m\,2\pi}\int\limits_{0}^{m\,2\pi} f(x)\,e^{-i\lambda x}\,dx = 0\,.$$

60. To elucidate further the concept of the Fourier series of an almost periodic function, we prove the following theorem.

SPECIAL THEOREM. $\underline{\text{Let the series}}\ \sum a_n e^{i\lambda_n x}$ (with distinct real exponents $\lambda_1, \lambda_2, \lambda_3, \ldots$ and non zero coefficients) $\underline{\text{be uniformly convergent for}}\ -\infty < x < \infty$ (or let it consist of only a finite number of terms), $\underline{\text{then this series is the}}$ $\underline{\text{Fourier series of its sum}}\ f(x)$.

Proof: We know already that the sum $f(x)$ is an almost periodic function. We must prove that

$$a(\lambda) = M\{f(x)\,e^{-i\lambda x}\} = M\big\{\sum a_n e^{i\lambda_n x}\,e^{-i\lambda x}\big\} = \begin{cases} a_n & \text{for}\ \lambda = \lambda_n \\ 0 & \text{for}\ \lambda \neq \lambda_n \end{cases}$$

This follows, however, from the fact that the series $\sum a_n e^{i\lambda_n x}$ $e^{-i\lambda x}$ converges uniformly for $-\infty < x < \infty$ (in case it has infinitely many terms). On that account, the mean value

process may be taken termwise.

    Corollary: Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots$ be a <u>completely</u> <u>arbitrary</u> (finite or infinite) sequence of real numbers. Then an almost periodic function can be constructed which has just these numbers Λ_n as Fourier exponents. (Thus, for instance, the sequence of exponents $\{\Lambda_n\}$ can have a finite accumulation point or even be everywhere dense in $-\infty < \lambda < \infty$.

    For example, let the coefficient $a_n \neq 0$ corresponding to the given exponent Λ_n be so chosen that $\sum |a_n|$ converges, (e.g., $a_n = \frac{1}{2^n}$), then the series $\sum a_n e^{i \Lambda_n x}$ will certainly be a Fourier series, namely, the Fourier series of its sum $f(x)$.

    61. Having made this interpolatory remark we resume our investigation. We consider again an <u>arbitrary</u> almost periodic function $f(x) \sim \sum A_n e^{i \Lambda_n x}$. We apply the inequality

(3) of §58 and we find by choosing $\lambda_n = \Lambda_n$, that for each N

$$\sum_{1}^{N} |A_n|^2 \leqq M\{|f(x)|^2\}.$$

<u>Thus the series</u> (with positive terms)

$$\sum |A_n|^2$$

<u>is convergent</u> (if it happens to contain infinitely many terms) and its sum is

$$\leqq M\{|f(x)|^2\}.$$

The fundamental fact will come out later that, exactly as in the case of purely periodic functions, the equality sign always holds in this relation, i.e., for an arbitrary almost periodic function, Parseval's equation holds:

$$\sum |A_n|^2 = M\{|f(x)|^2\}.$$

    62. Exactly as in the theory of Fourier series of purely periodic functions, we can also give this theorem another formulation, one to which it owes its central position in the theory. Consider the formula

$$M\left\{|f(x) - \sum_1^N A_n e^{i \Lambda_n x}|^2\right\} = M\{|f(x)|^2\} - \sum_1^N |A_n|^2,$$

which we obtain from formula (2)   of § 58 by choosing $\lambda_n = \Lambda_n$.
One sees from it, that when the Fourier series has only a
finite (say N_0) number of terms Parseval's equation re-
duces to

$$M\left\{|f(x) - \sum_1^{N_0} A_n e^{i \Lambda_n x}|^2\right\} = 0,$$

Moreover, in the general case, when the Fourier series con-
tains infinitely many terms, Parseval's equation reduces to

$$\lim_{N \to \infty} M\left\{|f(x) - \sum_1^N A_n e^{i \Lambda_n x}|^2\right\} = 0.$$

This last formulation states that the partial sums of the
Fourier series <u>converge in the mean to</u> $f(x)$.

63.   Entirely independently of the rather deep Parseval
equation, which settles the question of whether it is pos-
sible <u>with arbitrary accuracy</u> to approximate an almost per-
iodic function in the mean by means of the partial sums of
its Fourier series, there is one observation that we can
make. We can conclude immediately from formula (1) of § 57
that <u>if we wish to approximate an almost periodic function</u>

$f(x)$ <u>in the mean by a finite trigonometric sum</u> $\sum_1^N c_n e^{i \lambda_n x}$

<u>then we should choose the terms</u> $c_n e^{i \lambda_n x}$ <u>from among the terms</u>
<u>of the Fourier series of the function.</u> For, first of all,
the formula (1), shows that we should choose the exponents
λ_n from among the Fourier exponents, since we get a better
approximation (or a smaller mean-error) by omitting the
terms $c_n e^{i \lambda_n x}$ for which λ_n is not a Fourier exponent. Secondly,
formula (1) states that after the coefficients λ_n are chosen
from among the Fourier exponents Λ_n, the coefficients c_n
should be chosen as the corresponding Fourier coefficients A_n,
for it is clear that any other choice of c_n would yield a
poorer approximation.

CALCULATIONS WITH FOURIER SERIES

64. Theorems on the calculation with Fourier series of almost periodic functions are, for the most part, entirely analogous to the corresponding theorems for purely periodic functions.

From $f(x) \sim \sum A_n e^{i \Lambda_n x}$ it first follows that

(1) $$k f(x) \sim \sum k A_n \cdot e^{i \Lambda_n x}$$

(k an arbitrary complex constant). For

$$M\{k f(x) e^{-i \lambda x}\} = k M\{f(x) e^{-i \lambda x}\}.$$

(2) $$e^{i \Lambda x} f(x) \sim \sum A_n e^{i(\Lambda + \Lambda_n) x}.$$

For $$M\{e^{i \Lambda x} f(x) e^{-i \lambda x}\} = M\{f(x) e^{i(\Lambda - \lambda) x}\}.$$

(3) $$f(x + k) \sim \sum A_n e^{i \Lambda_n k} \cdot e^{i \Lambda_n x}$$

(k an arbitrary real constant). For

$$M\{f(x + k) e^{-i \lambda x}\} = e^{i \lambda k} M\{f(x + k) e^{-i \lambda (x+k)}\} = e^{i \lambda k} M\{f(x) e^{-i \lambda x}\}.$$

(4) $$\overline{f(x)} \sim \sum \overline{A_n} \, e^{-i \Lambda_n x}.$$

For
$$M\{\overline{f(x)} \, e^{-i \lambda x}\} = \overline{M\{f(x) e^{+i \lambda x}\}}.$$

From $f_1(x) \sim \sum A_n e^{i \Lambda_n x}$ and $f_2(x) \sim \sum B_n e^{i M_n x}$ it follows that

(5) $$f_1(x) + f_2(x) \sim \sum C_n e^{i N_n x},$$

where the last series is the result of the formal addition of the two series $\sum A_n e^{i \Lambda_n x}$ and $\sum B_n e^{i M_n x}$. (From formulas (1) and (5) we conclude that the Fourier series of the differences $f_1(x) - f_2(x)$ results from formal subtraction of the Fourier series of $f_1(x)$ and $f_2(x)$.

65. In contrast to the previously mentioned theorems, which are superficial, the following <u>general multiplication theorem</u> is quite deep.

The Fourier series of the product $f_1(x) f_2(x)$ is obtained by formal multiplication of the two series $\sum A_n e^{i \Lambda_n x}$ and $\sum B_n e^{i M_n x}$ Or, we have

(6) $$f_1(x) f_2(x) \sim \sum D_n e^{i \Pi_n x} \quad \text{with} \quad D_n = \sum_{\Lambda_p + M_q = \Pi_n} A_p B_q.$$

Or, to be exact, the exponent Π_n runs through the numbers of

the form $\Lambda_p + M_q$, and the corresponding coefficient D_n is given by the indicated sum, which, if it is an infinite series, converges absolutely (omitting, of course, the terms $D_n e^{i \Pi_n x}$ for which $D_n = 0$).

We shall get to the proof of the theorem later (§§ 74-76).

66. It is important for later applications to find the Fourier series of the "folded" function

$$g(x) = M\{f(x + t) \overline{f(t)}\}$$

where $f(x) \sim \sum A_n e^{i \Lambda_n x}$ is given. We know from § 54, that this function $g(x)$ is an almost periodic function. It will now be shown that

(7) $$g(x) \sim \sum |A_n|^2 e^{i \Lambda_n x},$$

or that for each λ , the mean value is

$$M\{g(x) e^{-i\lambda x}\} = |a(\lambda)|^2 = |M\{f(x) e^{-i\lambda x}\}|^2$$

For $\lambda = 0$ this is the relation already proved in § 54.
(*) $$M\{g(x)\} = |M\{f(x)\}|^2.$$

From this the more general relation follows immediately by substituting $f(x) e^{-i\lambda x}$ for $f(x)$ so that $g(x)$ is replaced by the folded function

$$M_t\{f(x + t) e^{-i\lambda(x+t)} \overline{f(t)} e^{i\lambda t}\} = g(x) e^{-i\lambda x}$$

and the relation (*) becomes the more general relation

$$M\{g(x) e^{-i\lambda x}\} = |M\{f(x) e^{-i\lambda x}\}|^2 .$$

Remarks: Exactly as in the case of purely periodic functions the formula (7) is a special case of a more general formula, according to which, if $f_1(x) \sim \sum A_n e^{i \Lambda_n x}$ and $f_2(x) \sim \sum B_n e^{i M_n x}$ are two arbitrary almost periodic functions, the function

$$g(x) = M_t\{f_1(x + t) f_2(t)\}$$

is also almost periodic and has as its Fourier series the series

(8) $$g(x) \sim \sum E_n e^{i P_n x}$$

where P_n takes on those values Λ_n for which $-\Lambda_n$ is one of the numbers M_n ; where $E_n = A_p B_q$, and where $P_n = \Lambda_p = -M_q$.

67. Let a series of almost periodic functions

$$f_m(x) \sim \sum A_n^{(m)} e^{i A_n^{(m)} x}$$

be given, which approaches a limit function $f(x)$ uniformly for all x as $m \to \infty$. Then

(9) $$f(x) = \lim_{m \to \infty} f_m(x) \sim \lim_{m \to \infty} \sum A_n^{(m)} e^{i A_n^{(m)} x}$$

This means that for a fixed λ, this limit equation holds:

$$M\{f(x) e^{-i\lambda x}\} = \lim_{m \to \infty} M\{f_m(x) e^{-i\lambda x}\}$$

(and in fact, holds uniformly for all λ)

The proof follows immediately from the fact that

$$M\{f(x) e^{-i\lambda x}\} - M\{f_m(x) e^{-i\lambda x}\} = M\{[f(x) - f_m(x)] e^{-i\lambda x}\} .$$

If an $\varepsilon > 0$ is given arbitrarily, then there exists a $M = M(\varepsilon)$ such that for $m > M$,

$$|f(x) - f_m(x)| \leqq \varepsilon \text{ for all } x.$$

From this it follows, however, that

$$|M\{f(x) e^{-i\lambda x}\} - M\{f_m(x) e^{-i\lambda x}\}| \leqq \varepsilon \quad \text{for. } m > M \text{ and all } \lambda.$$

68. In conclusion we take up the question of the indefinite integral $F(x)$ of an almost periodic function $f(x) \sim \sum A_n e^{i A_n x}$. Since the various indefinite integrals differ only by an additive constant, it is sufficient to consider only one of them, say

$$F(x) = \int_0^x f(\xi) d\xi .$$

We wish to investigate whether this function $F(x)$ is almost periodic, and if it is, to determine its Fourier series.

Here the analogy to purely periodic functions immediately breaks down. We should expect $F(x)$ to be almost periodic if and only if the Fourier series of $f(x)$ contains no constant term; that is, if $M\{f(x)\} = 0$. Moreover, in this case we should expect the formula

(10) $$F(x) \sim C + \sum \frac{A_n}{i A_n} e^{i A_n x} \qquad\qquad C = \text{const.}$$

to hold. It can be shown, however, that the condition $M\{f(x)\} = 0$ although necessary, is <u>no longer</u> sufficient

for the almost-periodicity of $F(x)$. If, however, $F(x)$ is almost periodic, (and an extremely simple necessary and sufficient condition for this will be given in §69), then the formula (10) is valid.

Proof: One sees immediately, that for the almost-periodicity of $F(x)$ it is necessary that

$$M\{f(x)\} = 0.$$

For if $M\{f(x)\} = c \neq 0$, i.e.,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\,dx = \lim_{T \to \infty} \frac{F(T)}{T} = c,$$

then $F(x)$ is not bounded. And furthermore, if the integral $F(x)$ is almost periodic, its Fourier series is given by the formula (10), as seen by integration by parts. In fact, for each $\lambda \neq 0$

$$M\{F(x)\,e^{-i\lambda x}\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(x)\,e^{-i\lambda x}\,dx$$

$$= \lim_{T \to \infty} \frac{1}{T} \left(\left[F(x) \frac{e^{-i\lambda x}}{-i\lambda} \right]_0^T + \frac{1}{i\lambda} \int_0^T f(x)\,e^{-i\lambda x}\,dx \right)$$

$$= -\frac{1}{i\lambda} \lim_{T \to \infty} \frac{F(T)}{T} e^{-i\lambda T} + \frac{1}{i\lambda} M\{f(x)\,e^{-i\lambda x}\} = \frac{1}{i\lambda} M\{f(x)\,e^{-i\lambda x}\},$$

since the presupposed almost-periodity of $F(x)$ implies the equation $M\{f(x)\}=0$, i.e., that $\lim_{T \to \infty} \frac{F(T)}{T} = 0$.

We must still show that the condition $M\{f(x)\} = 0$ is not sufficient for the almost-periodicity of $F(x)$. This follows, immediately, however, from what was just proven. For, if we choose a sequence of coefficients $A_n \neq 0$ and a sequence of distinct exponents $\Lambda_n \neq 0$, so that $\sum_1^\infty |A_n|$ converges and at the same time the series $\sum_1^\infty \left| \frac{A_n}{\Lambda_n} \right|^2$ diverges (e.g., $A_n = \Lambda_n = \frac{1}{2^n}$ then the uniformly convergent series $\sum A_n e^{i\Lambda_n x}$ obviously satisfies the condition $M\{f(x)\} = 0$ since $\Lambda_n \neq 0$ for all n, and, indeed, its Fourier series is the series

$\sum A_n e^{i\Lambda_n x}$.   Its integral $F(x)$ is, however, by no means

almost periodic, for then, we should have

$$F(x) \backsim C + \sum \frac{A_n}{i A_n} e^{i A_n x};$$

but the series on the right hand side is not the Fourier
series of an almost periodic function, since the series

$$\sum_{1}^{\infty} \left| \frac{A_n}{A_n} \right|^2 \text{ diverges.}$$

69.   We prove a very interesting theorem, which pro-
vides a necessary and sufficient condition for the almost
periodicity of the integral $F(x)$. This condition states
that the mere <u>boundedness</u> of $F(x)$ suffices for the almost-
periodicity of this function. That is, the following
theorem is valid.

THEOREM.   <u>For the almost-periodicity of the integral</u>
$F(x)$ <u>it is necessary and sufficient that</u> $F(x)$ <u>be bounded.</u>

<u>Proof:</u> Obviously $f(x)$ and also $F(x)$ may be taken as
real. By assumption the function $F(x)$ is bounded. We denote
its lower and upper bounds by k_1 and k_2 ; (where we can take
$k_1 < k_2$ since if $k_1 = k_2$ the function $F(x)$ is constant).

Let $\varepsilon > 0$ be given arbitrarily. The proof will   re-
sult from the fact <u>that a number</u> $\varepsilon_1 = \varepsilon_1(\varepsilon, f(x))$ <u>can be so deter-
mined that every translation number of the given function</u>
$f(x)$ <u>corresponding to</u> ε_1 <u>is also a translation number of</u>
$F(x)$ <u>corresponding to</u> ε .

To this end we choose two fixed values x_1 and x_2 such
that

$$F(x_1) < k_1 + \frac{\varepsilon}{6} \quad \text{and} \quad F(x_2) > k_2 - \frac{\varepsilon}{6}$$

We shall henceforth denote the smaller of them by ξ and the
distance $|x_2 - x_1|$ by d . We determine the length $l_0 = L\left(\frac{\varepsilon}{6d}\right)$
so   that in every interval of this length at least one
translation number τ of $f(x)$ exists, corresponding to $\varepsilon/6d$.
First we shall show that the oscillations of $F(x)$ display a
certain "regularity", namely, <u>that in each interval</u> $(\alpha, \alpha + L_0)$
<u>of length</u> $L_0 = l_0 + d$ <u>two values</u> z_1 <u>and</u> z_2 <u>can be found such
that</u>

(1) $F(z_1) < k_1 + \frac{\varepsilon}{2}$,   and $F(z_2) > k_2 - \frac{\varepsilon}{2}$

In fact, according to the definition of the length l_0 we can choose a translation number $\tau = \tau\left(\dfrac{\varepsilon}{6d}\right)$ of the function $f(x)$ so that the number $\xi + \tau$ falls in the interval $(\alpha, \alpha + l_0)$ and such that the two numbers $x_1 + \tau = z_1$ and $x_2 + \tau = z_2$ lie in the larger interval $(\alpha, \alpha + L_0)$. Then this relation is valid:

$$F(z_2) - F(z_1) = F(x_2) - F(x_1) + \int_{z_1}^{z_2} f(y)\, dy - \int_{x_1}^{x_2} f(y)\, dy = F(x_2) - F(x_1) + \int_{x_1}^{x_2} (f(y+\tau) - f(y))\, dy .$$

Accordingly

$$F(z_2) - F(z_1) \gtrsim F(x_2) - F(x_1) - d\,\frac{\varepsilon}{6d} > k_2 - k_1 - \frac{2\varepsilon}{6} - \frac{\varepsilon}{6} = k_2 - k_1 - \frac{\varepsilon}{2} .$$

But, by the definition of k_1 and k_2 , the inequality

$$F(z_2) - F(z_1) > k_2 - k_1 - \frac{\varepsilon}{2}$$

is possible only when $F(z_1)$ and $F(z_2)$ satisfy the desired inequalities (1) .

We now assert that the "small" number $\varepsilon_1 = \dfrac{\varepsilon}{2L_0}$ has the property mentioned above, and therefore that each "fine" translation number τ (i.e., one belonging to ε_1) is a translation number of the function $F(x)$ belonging to ε . This means that

(2) $$|F(x + \tau) - F(x)| \leq \varepsilon$$

for all x . To prove this, we use a simple artifice, and show that the two inequalities

and
(3a) $$F(x + \tau) - F(x) \geq -\varepsilon$$

(3b) $$F(x + \tau) - F(x) \leq \varepsilon$$

hold.

(a)   For the proof of the inequality (3a) we choose a number z_1 in the interval $(x, x + L_0)$ where x is arbitrarily given and z_1 satisfies the condition $F(z_1) < k_1 + \dfrac{\varepsilon}{2}$. Then we have

$$F(x + \tau) - F(x) = F(z_1 + \tau) - F(z_1) + \int_{x}^{x+\tau} f(y)\, dy - \int_{z_1}^{z_1+\tau} f(y)\, dy$$

$$= F(z_1 + \tau) - F(z_1) + \int_{x}^{z_1} f(y)\, dy - \int_{x+\tau}^{z_1+\tau} f(y)\, dy$$

$$> k_1 - \left(k_1 + \frac{\varepsilon}{2}\right) - \left| \int_x^{z_1} (f(y+\tau) - f(y))\, dy \right| > -\frac{\varepsilon}{2} - L_0 \frac{\varepsilon}{2 L_0} = -\varepsilon.$$

(b)   The proof of the inequality (3b) follows similarly, except that this time we use a point z_2 in the interval

$(x, x + L_0)$ for which $F(z_2) > k_2 - \frac{\varepsilon}{2}$

We then obtain analogously

$$F(x+\tau) - F(x) = F(z_2 + \tau) - F(z_2) + \int_x^{z_1} f(y)\, dy - \int_{x+\tau}^{z_2+\tau} f(y)\, dy$$

$$< k_2 - \left(k_2 - \frac{\varepsilon}{2}\right) + \left| \int_x^{z_2} (f(y+\tau) - f(y))\, dy \right| < \frac{\varepsilon}{2} + L_0 \frac{\varepsilon}{2 L_0} = \varepsilon.$$

This concludes the proof.

THE UNIQUENESS THEOREM
ITS EQUIVALENCE WITH PARSEVAL'S EQUATION

70.   As we shall later show for almost periodic functions, an almost periodic function is always uniquely determined by its Fourier series, or, that two distinct functions $f_1(x)$ and $f_2(x)$ always determine two distinct Fourier series. Exactly as in the case of purely periodic functions this uniqueness theorem can be expressed in another form. To this end we consider the function $f(x) = f_1(x) - f_2(x)$. As we noticed earlier, the Fourier series of this function $f(x)$ results from formal subtraction of the two Fourier series corresponding respectively to $f_1(x)$ and $f_2(x)$. Thus, the two functions $f_1(x)$ and $f_2(x)$ have the same Fourier series if, and only if, the Fourier series corresponding to $f(x)$ has no non-zero terms, or

$$a(\lambda) = M\{f(x)\, e^{-i\lambda x}\} = 0$$

for all λ. Thus the uniqueness theorem can be formulated as follows.: There is no almost periodic function $f(x)$, which does not identically vanish, for which $a(\lambda) = 0$ for all λ.

Using a terminology similar to that of harmonic oscillation e^{inx} with respect to the non-enumerable totality of all oscillations $e^{i\lambda x}$, we can express the uniqueness theorem in another form. Thus, we say that there exists no almost periodic function $f(x)$, not identically zero, such

that the orthogonality relation

$$M\{f(x)e^{-i\lambda x}\} = 0$$

is satisfied for each λ . Thus we also say that the sys-
tem $\{e^{i\lambda x}\}$ orthogonal in the interval $-\infty < x < \infty$ is com-
plete in the class of almost periodic functions. (We could
have also restricted our considerations to normalized func-
tions at this point for we shall show in §72 that if an arb-
itrary almost periodic function does not vanish identically,
we can multiply it by a suitably chosen constant
and make the mean value of its square unity.)

It should be expressly mentioned, however, that it
would not have been permissible to omit the words "in the
class of almost periodic functions." In fact, there exist
some rather simple functions $f(x)$, that are not almost per-
iodic such that, for each λ

$$\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} f(x)\, e^{-i\lambda x}\, dx = 0$$

is satisfied, (even uniformly with respect to a), without
the function vanishing identically. This happens, not only
in such trivial cases as, those for example, in which $f(x) \to 0$
as $|x| \to \infty$, but also, as can be seen by simple approxima-
tions, for functions like e^{ix^2} and $e^{i\sqrt{x}}$. The first of
these functions has very "rapid" oscillations and the second
has very "slow" oscillations as $|x| \to \infty$ while the absolute
value of each is even constant (in fact unity).

Example: If a purely periodic function $f(x)$ of period
2π is thought of as almost periodic, then we know (accord-
ing to §59) that its Fourier exponents Λ_n are all integral.
From the uniqueness theorem it would follow conversely that

an almost periodic function $f(x) \sim \sum A_n e^{i\Lambda_n x}$ with integral ex-
ponents is necessarily a purely periodic function of period
2π . In fact the Fourier series of the function $f(x + 2\pi)$,
is determined from the Fourier series of $f(x)$ by applying
formula (3) of §64, or by the mere substitution of $x + 2\pi$
for x in that Fourier series. Thus we find that the new

series is identical with the original series $\sum A_n e^{i\Lambda_n x}$.

71. The uniqueness theorem is a really deep theorem,

whose proof we shall defer until later. We shall, in the
meantime, content ourselves with showing that it is fully
equivalent to the other fundamental theorem of the theory,
namely Parseval's equation, which states that <u>for an arbi-</u>

<u>trary almost periodic function</u> $f(x) \sim \sum A_n e^{i A_n x}$ <u>the relation</u>

$$\sum |A_n|^2 = M\{|f(x)|^2\}$$

holds. The proof of this equivalence is completely analog-
ous to the corresponding proof for periodic functions.

    1. <u>The uniqueness theorem follows from Parseval's</u>
<u>equation.</u>

    <u>Proof</u>: Let $f(x)$ be an almost periodic function, whose
Fourier series has no terms whatsoever. Then from Parseval's

equation $M\{|f(x)|^2\} = \sum |A_n|^2$, it follows that $M\{|f(x)|^2\} = 0$.
Our problem is to **show** on the basis of $M\{|f(x)|^2\} = 0$ that

$f(x)$ vanishes identically. For purely periodic functions the
corresponding conclusion is trivial, since, in fact, the
mean value of a non negative continuous function over a fin-
ite interval can vanish only if the function vanishes iden-
tically. For almost periodic functions, where we are taking
a mean value over an <u>infinite</u> interval the conclusion is
(fortunately) still true, although not immediate. We shall
defer the proof to the next paragraph in order to prove for
later use some further theorems in the same connection.

    2. <u>Parseval's theorem follows from the uniqueness</u>
<u>theorem.</u>

    <u>Proof</u>: Let $f(x) \sim \sum A_n e^{i A_n x}$ be an arbitrary almost per-
iodic function. We form the following function (which, by
formula (7) in § 66 is itself almost periodic):

$$g(x) = M\{f(x+t)\overline{f(t)}\} \sim \sum |A_n|^2 e^{i A_n x}.$$

Since the series $\sum |A_n|^2$ converges (with a sum $\leq M\{|f(x)|^2\}$,

by § 6I), then the last term $\sum |A_n|^2 e^{i A_n x}$ is <u>uniformly</u>

<u>convergent for all</u> x (when it contains, as it does in
general, infinitely many terms). Thus (by the theorem of
§ 60) the latter series is the Fourier series of its (eo
ipso) almost periodic sum $s(x)$. The two almost periodic
functions $g(x)$ and $s(x)$ accordingly have the same Fourier

series (namely $\sum |A_n|^2 e^{i A_n x}$) and therefore, by the uniqueness

theorem, $g(x) = s(x)$, i.e.,
$$g(x) = \sum |A_n|^2 e^{i \cdot \Lambda_n x}.$$

If, in this equation, we choose in particular $x = 0$, we get
$$g(0) = M_t\{f(t)\,\overline{f(t)}\} = M\{|f(t)|^2\} = \sum |A_n|^2,$$

or Parseval's equation.

72. From the preceeding section we still need to prove the following theorem:

THEOREM. An almost periodic function $f(x)$, for which $M\{|f(x)|^2\} = 0$, must vanish identically, i.e., for all x

Proof: We give an indirect proof, i.e., we assume that $f(x)$ is not identically zero, and then we show that $M\{|f(x)|^2\} > 0$. By our assumption, a positive number α must exist such that in some point, say x_0, the inequality $|f(x_0)| \geqq \alpha$ holds. Now, by a previous remark (see §47) there exists a length corresponding to the number $\varepsilon = \frac{\alpha}{2}$. Call this length $L(\varepsilon) = L\left(\frac{\alpha}{2}\right)$. There exists a positive number $\delta(\varepsilon) = \delta\left(\frac{\alpha}{2}\right)$, such that each interval of length L contains a complete interval of length δ whose points τ are all translation numbers $\tau_f(\varepsilon) = \tau_f\left(\frac{\alpha}{2}\right)$ and thus satisfy the equation:
$$|f(x_0 + \tau)| \geqq |f(x_0)| - |f(x_0 + \tau) - f(x_0)| \geqq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$
Thus the integral of $|f(x)|^2$ extended over an arbitrary interval of length $L\left(\frac{\alpha}{2}\right)$ is surely $\geqq \delta\left(\frac{\alpha}{2}\right) \cdot \left(\frac{\alpha}{2}\right)^2$, and the mean value $M\{|f(x)|^2\}$ satisfies the inequality
$$M\{|f(x)|^2\} = \lim_{m \to \infty} \frac{1}{m L\left(\frac{\alpha}{2}\right)} \int_0^{m L\left(\frac{\alpha}{2}\right)} |f(x)|^2\, dx \geqq \frac{\delta\left(\frac{\alpha}{2}\right) \cdot \left(\frac{\alpha}{2}\right)^2}{L\left(\frac{\alpha}{2}\right)},$$

and therefore it is positive, contrary to assumption.

Remarks: We say that a set $\{\varphi(x)\}$ of almost periodic functions is majorized by the almost periodic function $f(x)$, or that it is majorizable with majorant $f(x)$, if each translation number $\tau = \tau_f(\varepsilon)$ of $f(x)$ corresponding to an arbi-

trary number $\varepsilon > 0$ is at the same time a translation number $\tau_\varphi(\varepsilon)$ corresponding to ε for each function $\varphi(x)$ of our set. Then the above proof shows the truth of the following assertion:

Let $\{\varphi(x)\}$ be a set of almost periodic functions which is majorized by the almost periodic function $f(x)$. Then there exists for each $\alpha > 0$ a number $\beta = \beta(\alpha) > 0$, such that the inequality

$$M\{|\varphi(x)|^2\} \geqq \beta$$

holds for every function $\varphi(x)$ of our set for which

$$\underset{-\infty < x < \infty}{\text{upper limit}} |\varphi(x)| > \alpha.$$

In fact, we can take for $\beta = \beta(\alpha)$ the value just found,

$$\beta = \frac{\delta\left(\frac{\alpha}{2}\right) \cdot \left(\frac{\alpha}{2}\right)^2}{L\left(\frac{\alpha}{2}\right)},$$

where $L\left(\frac{\alpha}{2}\right)$ and $\delta\left(\frac{\alpha}{2}\right)$ correspond to $f(x)$, the majorant of the set; and accordingly could be used all the more for the individual functions $\varphi(x)$ of the set.

73. An immediate corollary of the last remark is the following theorem: Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ be a majorizable sequence of almost periodic functions (majorized by an almost periodic function $f(x)$), , and let it only be assumed that

$$M\{|\varphi_n(x)|^2\} \to 0 \quad \text{for} \quad n \to \infty;$$

Then $\varphi_n(x)$ converges to 0 as $n \to \infty$, and indeed uniformly in the interval $-\infty < x < \infty$, i.e.,

$$\underset{-\infty < x < \infty}{\text{up}} |\varphi_n(x)| \to 0 \quad \text{for} \quad n \to \infty.$$

In fact, for an arbitrarily given $\alpha > 0$ we can find a number $\beta = \beta(\alpha) > 0$ such that

$$\underset{-\infty < x < \infty}{\text{upper limit}} |\varphi_n(x)| \leqq \alpha, \quad \text{as long as} \quad M\{|\varphi_n(x)|^2\} < \beta$$

and the last inequality $M\{|\varphi_n(x)|^2\} < \beta$ is surely satisfied for all sufficiently large values of n.

We shall later apply these remarks in the following form: Let $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ be a sequence of almost periodic functions which are majorized by the almost

periodic function $f(x)$. Then it follows that $f_n(x)$ con-
verges uniformly as $n \to \infty$ in case it is known that $f_n(x)$
approaches $f(x)$ in the mean as $n \to \infty$. i.e., whenever

$$M\{|f(x) - f_n(x)|^2\} \to 0 \quad \text{for} \quad n \to \infty .$$

As proof we have only to set $f(x) - f_n(x) = \varphi_n(x)$. Then
the sequence $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ is obviously majoriz-
able, namely with majorant $2f(x)$. (For $\tau_{2f}(\varepsilon) = \tau_f(\varepsilon/2) = \tau_{f_n}$
$(\varepsilon/2) = \tau_{f-f_n}(\varepsilon))$, and thus from the hypothesis $M\{|\varphi_n(x)|^2\} \to 0$
as $n \to \infty$, it follows that $\varphi_n(x)$ converges uniformly to zero
as $n \to \infty$.

THE MULTIPLICATION THEOREM

74. Before we prove the uniqueness theorem and with it
Parseval's equation, we wish to give a proof of the multi-
plication theorem (6) in §65 on the basis of these last two
theorems. Specifically, we wish to show that the multipli-
cation theorem is completely equivalent with Parseval's
equation.

The general multiplication theorem states: If $f_1(x) \sim$

$\sum A_n e^{i \Lambda_n x}$ and $f_2(x) \sim \sum B_n e^{i M_n x}$ are two arbitrary almost per-

iodic functions then

(6) $$f_1(x) f_2(x) \sim \sum D_n e^{i \Pi_n x} , \quad D_n = \sum_{\Lambda_p + M_q = \Pi_n} A_p B_q .$$

As already mentioned, this means, that the exponent Π_n runs
through all numbers of the form $\Lambda_p + M_q$ and, further, that
the corresponding coefficient D_n is given by the indicated
sum where the series, if it has an infinite number of terms,
is absolutely convergent. (Naturally we omit terms $D_n e^{i \Pi_n x}$
with $D_n = 0$.)

It is a matter of showing that for every real ν

$$M\{f_1(x) f_2(x) e^{-i \nu x}\} = \sum_{\Lambda_p + M_q = \nu} A_p B_q$$

(where the right side is taken to be zero for an empty sum),
and of showing also, that the right hand series converges
absolutely when it has infinitely many terms. We note at
this point that the last part of the theorem is trivial be-

cause of the elementary inequality $|ab| \leq \tfrac{1}{2}(|a|^2 + |b|^2)$. By this

inequality the series $\sum_{\Lambda_p + M_q = \nu} |A_p B_q|$ is obviously majorized by the

series $\frac{1}{2} \sum\limits_{A_p + M_q = \nu} (|A_p|^2 + |B_q|^2)$. The convergence of this latter series, in turn, follows immediately from the convergence of the two series $\sum |A_n|^2$ and $\sum |B_n|^2$, since (because of the condition $A_p + M_q = \nu$) the same coefficient A_n or B_n never enters in two different products of the form $A_p B_q$.

We notice further that it suffices to consider the special case $\nu = 0$ where we work with the constant term of the Fourier series or with the equation

(*) $M\{f_1(x) f_2(x)\} = \sum\limits_{A_p + M_q = 0} A_p B_q .$

If this equation is indeed valid, we can simply replace $f_2(x)$ by the function $f_2(x) e^{-i\nu x} \sim \sum B_n e^{i(M_n - \nu)x}$ and obtain the more general equation

$$M\{f_1(x) f_2(x) e^{-i\nu x}\} = \sum\limits_{A_p + (M_q - \nu) = 0} A_p B_q = \sum\limits_{A_p + M_q = \nu} A_p B_q .$$

75. We now go on to the proof itself and show first
1. Parseval's equation follows from the multiplication theorem.

If one makes the special choice $f_2(x) = \overline{f_1(x)} \sim \sum \overline{A_n} e^{-iA_n x}$ then from (*) it follows that

$$M\{f_1(x) \overline{f_1(x)}\} = \sum\limits_{A_p + (-A_q) = 0} A_p \overline{A_q} ,$$

which is nothing but Parseval's equation

$$M\{|f_1(x)|^2\} = \sum |A_n|^2 .$$

2. The Multiplication theorem follows from Parseval's equation.

The proof depends on the same simple trick as the one we used in the theory of purely periodic functions, namely, the use of the elementary identity.

$$u v = \tfrac{1}{4}\{|u + \bar{v}|^2 - |u - \bar{v}|^2 + i|u + i\bar{v}|^2 - i|u - i\bar{v}|^2\}.$$

If we apply this to the product $f_1(x) f_2(x)$, it yields

$$M\{f_1 f_2\} = \tfrac{1}{4}[M\{|f_1 + \overline{f_2}|^2\} - M\{|f_1 - \overline{f_2}|^2\} + i M\{|f_1 + i\overline{f_2}|^2\} - i M\{|f_1 - i\overline{f_2}|^2\}]$$

If we now apply Parseval's equation to the functions $f_1(x) + \overline{f_2(x)}$, $f_1(x) - \overline{f_2(x)}$, $f_1(x) + if_2(x)$ and $f_1(x) - i\overline{f_2(x)}$, and using their Fourier series which can be obtained from formulas (1), (4) and (5) of §64, and then momentarily setting

$$a(\lambda) = M\{f_1(x) e^{-i\lambda x}\} \quad \text{and} \quad b(\lambda) = M\{f_2(x) e^{-i\lambda x}\}$$

we find that the right hand member of the above relation be-
comes

$$= \tfrac{1}{4}[\sum |a(\lambda) + \overline{b(-\lambda)}|^2 - \sum |a(\lambda) - \overline{b(-\lambda)}|^2 + i \sum |a(\lambda) + i\overline{b(-\lambda)}|^2$$
$$- i \sum |a(\lambda) - i\overline{b(-\lambda)}|^2]$$

where each summation is over those values of λ for which
the summand is positive.

Here we can apply the same identity, only in the re-
verse direction and obtain

$$M\{f_1 f_2\} = \sum a(\lambda) b(-\lambda),$$

where the summation is interpreted as above. This, however,
is just the desired relation (*).

76. Finally we notice that we can still prove the
multiplication theorem exactly as for purely periodic func-
tions, that is, without using the above artifice but rather
by starting with the <u>uniqueness</u> theorem. We have only to
draw upon formula (8) of § 66 according to which the (almost
periodic) function

$$g(x) = M_t \{f_1(x+t) f_2(t)\}$$

has the Fourier series

$$g(x) \sim \sum E_n e^{i P_n x}$$

where P_n takes on the values Λ_n for which $-\Lambda_n$ occurs among
the set of numbers M_n and where $E_n = A_p B_q$, if $P_n = \Lambda_p = -M_q$.

The series converges uniformly, the series $\sum E_n$ being

the absolutely convergent series $\sum_{\Lambda_p + M_q = 0} A_p B_q$. Thus the

sum of the Fourier series must be an almost periodic func-

tion $s(x)$, whose Fourier series is exactly the series

$\sum E_n e^{i P_n x}$, and which must coincide, by the uniqueness
theorem, with the function $g(x)$. Now, in the relation just
derived

$$g(x) = \sum E_n e^{i P_n x}$$

we substitute the value $x = 0$ for x and obtain

$$g(0) = M\{f_1(t) f_2(t)\} = \sum E_n = \sum_{\Lambda_p + M_q = 0} A_p B_q ,$$

which is the desired relation(*).

INTRODUCTORY REMARKS
TO THE PROOF OF THE TWO
FUNDAMENTAL THEOREMS

77. We have now arrived at the most difficult part of
the theory, namely, the proof of the two (equivalent) Funda-
mental theorems, the uniqueness theorem and Parseval's
equation. Originally, the author proved the two fundamental
theorems by proving the Parseval Equation

$$M\{|f(x)|^2\} = \sum |A_n|^2$$

since he did not know at that time of the derivation of this
equation from the uniqueness theorem. After the complete
equivalence of the two Fundamental theorems has been shown it
would naturally be convenient to give the proof on the basis
of the uniqueness theorem. By what we already know this
amounts to proving Parseval's equation only for the very
special case where the Fourier series of the almost periodic
function $f(x)$ has no terms whatsoever, and where we are deal-
ing with the equation

(*) $$M\{|f(x)|^2\} = 0.$$

From this equation, it follows, indeed, as was shown in § 72
that $f(x)$ vanishes identically.
     The equation (*), i.e., the uniqueness theorem, can be
proved in several ways. Naturally one such proof results
from specializing the author's original proof of the general·
Parseval equation. This proof, however, is very involved.
Another proof, which rests on the same basic ideas was given
by de la Vallée Poussin. This proof will be given here. In
order that the ingenious and rather subtle reasoning of the
proof may stand out as clearly as possible, we shall first
briefly sketch the starting point of the original proof. We
shall then have the opportunity of seeing where the prin-
ciple simplifications of the new proof lie.

     78. Let an almost periodic function $f(x)$ be given for
which

$$a(\lambda) = M\{f(x) e^{-i\lambda x}\} = 0$$

for each λ . Then we are to show that

$$M\{|f(x)|^2\} = \lim_{T \to \infty} \frac{1}{T}\int\limits_0^T |f(x)|^2\, dx = 0.$$

Since the assumption $a(\lambda) = 0$ is put to use only at the con-
clusion, it will be convenient not to make this assumption
for the time being. Our considerations refer, then, to an
arbitrary almost periodic function $f(x)$.

The idea of the proof consists in considering the
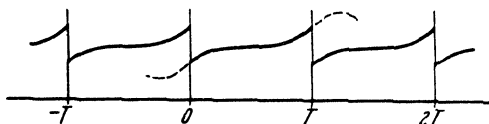**purely periodic function** $F(x) = F_T(x)$ with period T

Fig. 8.

which is equal to the given function $f(x)$ **in the interval**
$0 < x < T$. Thus

$$F_T(x) = f(x) \quad \textbf{for} \quad 0 < x < T, \quad F_T(x + T) = F_T(x).$$

We then apply Parseval's equation to the function. The
function $F_T(x)$ need not be continuous, but will, in general,
have a saltus at each of the points mT $(m = 0, \pm 1, \pm 2, \ldots)$.
This does not matter for we have considered functions of
this type before, in §§ 37-38, in anticipation, in fact, of
just this application. We have shown, among other results,
that Parseval's equation is valid for these functions. Let
us write the Fourier series of $F_T(x)$ in the form

$$F_T(x) \sim \sum_{-\infty}^{\infty} \alpha_n\, e^{i\frac{2\pi}{T}nx}.$$

(Here not only the exponents $\frac{2\pi}{T}n$, but also the coefficients

$$\alpha_n = \frac{1}{T}\int\limits_0^T F_T(x)\, e^{-i\frac{2\pi}{T}nx}\, dx = \frac{1}{T}\int\limits_0^T f(x)\, e^{-i\frac{2\pi}{T}nx}\, dx$$

depend on T). There follows the relation

$$\sum_{-\infty}^{\infty} |\alpha_n|^2 = \frac{1}{T}\int\limits_0^T |F_T(x)|^2\, dx = \frac{1}{T}\int\limits_0^T |f(x)|^2\, dx.$$

Now in this relation the right hand member approaches $M\{|f(x)|^2\}$; as $T \to \infty$, and therefore the same is true of the left hand member; i.e., for the mean value $M\{|f(x)|^2\}$ we now have the new expression

$$M\{|f(x)|^2\} = \lim_{T \to \infty} \sum_{-\infty}^{\infty} |\alpha_n|^2.$$

We have been considering an arbitrary almost periodic function; now we can go back to the original problem. This problem can be expressed as follows: <u>We are to show that, from the assumption</u> $a(\lambda) = 0$, <u>the limit equation</u>

$$\lim_{T \to \infty} \sum_{-\infty}^{\infty} |\alpha_n|^2 = 0.$$

<u>follows</u> or, in other words, that for large T the <u>sum of squares</u> $\sum_{-\infty}^{\infty} |\alpha_n|^2$ will turn out to be very small.

79. This is this way the uniqueness theorem can be proved. The direct proof of the given limit equation is, however, very difficult. On the other hand, as we shall very soon show, it is not particularly difficult to show that for large T the <u>upper bound</u> of the members $|\alpha_n|$ is very small, i.e., that <u>from the assumption</u> $a(\lambda) = 0$ there follows the

(*) $\lim_{\substack{T \to \infty \\ -\infty < n < \infty}} \text{upper bound } |\alpha_n| = 0$.

It is very interesting to realize that this much weaker limit equation yields the uniqueness theorem. To do this, however, we must alter the starting point of the proof as did de la Vallée Poussin. We must compare not the two functions $f(x)$ and $F_T(x)$, but the two <u>folded functions</u>

$$g(t) = M_t\{f(x+t)\,\overline{f(t)}\} \quad \text{and} \quad G_T(x) = \frac{1}{T}\int_0^T F_T(x+t)\,\overline{F_T(t)}\,dt.$$

Before examining this matter in more detail, we first give the proof of the limit equation (*).

<div align="center">PRELIMINARIES FOR THE PROOF OF
THE UNIQUENESS THEOREM</div>

80. First we wish to derive some lemmas.

THEOREM I. <u>Let</u> $f(x)$ <u>be an arbitrary almost periodic function and</u> ε <u>an arbitrary positive quantity</u>. <u>Then numb-</u>

ers Λ and T_0 can be chosen so large that for each $|\lambda| > \Lambda$ and each $T > T_0$

$$\left| \frac{1}{T} \int_0^T f(x) e^{-i\lambda x} dx \right| < \varepsilon.$$

Proof: From what precedes, it is clear that we can so choose the constants that for $|\lambda| > \Lambda$ this inequality holds:

$$|a(\lambda)| = \left| \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x) e^{-i\lambda x} dx \right| < \varepsilon.$$

For, there exists for a given $\varepsilon > 0$ only a finite number of numbers λ for which $|a(\lambda)| \geqq \varepsilon$. From this the inequality in question follows but with T_0 dependent on λ . To show that T_0 can be chosen independently of λ , we must proceed differently. This proof is based on the uniform continuity and boundedness of $f(x)$. In fact we even could choose $T_0 = 1$.

First we notice that for a given $\lambda \neq 0$ the function $e^{-i\lambda x}$ is periodic with period $2\pi/|\lambda|$, and that, further, the integral of this function over an arbitrary interval of length $2\pi/|\lambda|$ equals zero. From this we get for any x_1 , the equation

$$\int_{x_1}^{x_1 + \frac{2\pi}{|\lambda|}} f(x) e^{-i\lambda x} dx = \int_{x_1}^{x_1 + \frac{2\pi}{|\lambda|}} (f(x) - f(x_1)) e^{-i\lambda x} dx .$$

We let $\omega(\delta)$ denote the upper limit of $|f(x_1) - f(x_2)|$ for $|x_1 - x_2| \leqq \delta$, for a given $\delta > 0$, then we have

$$\left| \int_{x_1}^{x_1 + \frac{2\pi}{|\lambda|}} f(x) e^{-i\lambda x} dx \right| \leqq \omega\left(\frac{2\pi}{|\lambda|}\right) \cdot \frac{2\pi}{|\lambda|} .$$

Let an arbitrary $T > 0$ be given and let $T = n \frac{2\pi}{|\lambda|} + \alpha$, where n is an integer and $0 \leqq \alpha < \frac{2\pi}{|\lambda|}$. Then

$$\int_0^T f(x) e^{-i\lambda x} dx = \int_0^{\frac{2\pi}{|\lambda|}} + \int_{\frac{2\pi}{|\lambda|}}^{\frac{4\pi}{|\lambda|}} + \cdots + \int_{(n-1)\frac{2\pi}{|\lambda|}}^{n\frac{2\pi}{|\lambda|}} + \int_{n\frac{2\pi}{|\lambda|}}^T f(x) e^{-i\lambda x} dx ,$$

and thus, letting Γ denote the upper bound of $|f(x)|$, we have

$$\left|\int_0^T f(x)e^{-i\lambda x}dx\right| \leqq n\,\omega\left(\frac{2\pi}{|\lambda|}\right)\frac{2\pi}{|\lambda|} + \Gamma\alpha \leqq T\cdot\omega\left(\frac{2\pi}{|\lambda|}\right) + \Gamma\frac{2\pi}{|\lambda|}.$$

Thus we find finally that for $T > T_0 = 1$

$$\left|\frac{1}{T}\int_0^T f(x)e^{-i\lambda x}dx\right| \leqq \omega\left(\frac{2\pi}{|\lambda|}\right) + \frac{\Gamma\cdot\frac{2\pi}{|\lambda|}}{T} \leqq \omega\left(\frac{2\pi}{|\lambda|}\right) + \Gamma\frac{2\pi}{|\lambda|}.$$

From here on the proof is obvious. For from the uniform continuity of $f(x)$, it follows that $\omega(\delta)\to 0$ as $\delta\to 0$. Thus a $\Lambda = \Lambda(\varepsilon)$ can so be chosen that for $|\lambda| > \Lambda$ the following inequality holds:

$$\omega\left(\frac{2\pi}{|\lambda|}\right) + \Gamma\frac{2\pi}{|\lambda|} < \varepsilon .$$

THEOREM II. <u>Let</u> $f(x)$ <u>be an almost periodic function and let</u> λ_0 <u>be a real number for which</u> $a(\lambda_0) = M\{f(x)e^{-i\lambda_0 x}\} = 0$ <u>(i.e.,</u> λ_0 <u>is not one of the fourier exponents of</u> $f(x)$ <u>,)</u> · <u>then for each given</u> $\varepsilon > 0$ <u>the positive numbers</u> δ <u>and</u> T_0 <u>can be determined so that for each</u> λ <u>of the interval</u> $(\lambda_0 - \delta,$ $\lambda_0 + \delta)$ <u>and each</u> $T > T_0$

$$\left|\frac{1}{T}\int_0^T f(x)e^{-i\lambda x}dx\right| < \varepsilon .$$

<u>Proof</u>: It is clear from what precedes, that we can choose the constant δ so that for $|\lambda - \lambda_0| < \delta$ we have the equation

$$|a(\lambda)| = \left|\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)e^{-i\lambda x}dx\right| < \varepsilon .$$

From this the equation in question follows except that again T_0 depends on λ . To show that T_0 can be chosen independently of λ , we must proceed differently. The proof depends on the improved mean value theorem for almost periodic functions.

We notice first, that without loss of generality we can

assume that $\lambda_0 = 0$. (For otherwise we need only consider
the function $f_1(x) = f(x)\,e^{-i\lambda_0 x}$ in place of $f(x)$. On hypo-
thesis is, then, that

$$a(0) = M\{f(x)\} = 0 ,$$

and we have to show that

$$\left| \frac{1}{T} \int_0^T f(x)\,e^{-i\lambda x}\,dx \right| < \varepsilon \ \text{for} \ |\lambda| < \delta, \quad T > T_0 .$$

In this formulation the theorem has a marked similarity
to Theorem I, but there we were concerned with the very
"rapid" oscillation $e^{-i\lambda x}$ (namely with the large values of
$|\lambda|$ and the proof rested on the fact that factor $f(x)$
hardly changes in the course of a full oscillation or $e^{-i\lambda x}$.
In the theorem to be proved now the situation is exactly re-
versed; here we are concerned with very "slow" oscillations
$e^{-i\lambda x}$ (where the values of $|\lambda|$ are small, and the proof
will depend on the fact that the factor $e^{-i\lambda x}$ changes too
slowly to cancel out the oscillations of $f(x)$ that are im-
plied in the condition $M\{f(x)\} = 0$. The idea is carried
out as follows:

By the improved mean value theorem of §52, the mean
value
$$\frac{1}{T} \int_a^{a+T} f(x)\,dx$$

converges to $M\{f(x)\}$, uniformly in a , as $T \to \infty$. Thus in
this case, it converges to zero and we can first choose T_0
so large that for $H > T_0$ and each a

$$\left| \frac{1}{H} \int_a^{a+H} f(x)\,dx \right| < \frac{\varepsilon}{2} .$$

This T_0 will be the T_0 of our theorem. Let Γ now be the
upper limit of $|f(x)|$ and let us now choose the number $\eta > 0$
so small that

$$\text{upper limit}_{|x_1 - x_2| < \eta} |e^{ix_1} - e^{ix_2}| < \frac{\varepsilon}{2\Gamma}$$

(which is possible because of the uniform continuity of e^{ix}).
If we choose $\delta = \eta/2T_0$ then for each $|\lambda| < \delta$.

$$\underset{|x_1-x_2|<2T_0}{\text{upper limit}}\; \left|e^{-i\lambda x_1}-e^{-i\lambda x_2}\right| < \frac{\varepsilon}{2\Gamma}\;.$$

This δ will be the δ of our theorem. Now it is always true that

$$\frac{1}{H}\int_a^{a+H} f(x)e^{-i\lambda x}\,dx = \frac{1}{H}\int_a^{a+H} f(x)e^{-i\lambda a}\,dx + \frac{1}{H}\int_a^{a+H} f(x)\left(e^{-i\lambda x}-e^{-i\lambda a}\right)dx.$$

Thus it follows from the above that for $|\lambda| < \delta$ for $T_0 < H \leqq 2T_0$ and, for each a the following inequality holds:

$$\left|\frac{1}{H}\int_a^{a+H} f(x)e^{-i\lambda x}\,dx\right| < \frac{\varepsilon}{2}\cdot\left|e^{-i\lambda a}\right| + \Gamma\cdot\frac{\varepsilon}{2\Gamma} = \varepsilon\;.$$

The proof of our theorem is now nearly complete. Let T be an arbitrary number $>T_0$; we can then choose a posi-
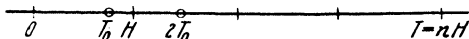

Fig. 9.

tive integer n such that the number $T/n = H$ lies in the interval $T_0 < H \leqq 2T_0$. (We have only to determine that integer $k \geqq 0$ for which $2^k T_0 < T \leqq 2^{k+1}T_0$, and then the number in question can be chosen as $n = 2^k$). Then the mean value

$$\frac{1}{T}\int_0^T f(x)e^{-i\lambda x}\,dx$$

is the arithmetic mean of the n mean value

$$\frac{1}{H}\int_{(\nu-1)H}^{\nu H} f(x)e^{-i\lambda x}\,dx, \qquad\qquad \nu = 1, 2, \ldots, n$$

and it is, therefore, numerically smaller than ε for $|\lambda| < \delta$.

81. With the help of Theorems I and II, we now prove the following:

PRINCIPAL LEMMA. <u>Let</u> $f(x)$ <u>be an almost periodic function for which</u> $a(\lambda) = M\{f(x)e^{-i\lambda x}\} = 0$ <u>for each and every</u> λ. <u>Then the limit equation</u>

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\, e^{-i\lambda x}\, dx = 0$$

holds uniformly for $-\infty < \lambda < \infty$, i.e., to a given $\varepsilon > 0$ a $T_0 = T_0(\varepsilon)$ can be chosen so that for $T > T_0$ and all λ

$$\left| \frac{1}{T} \int_0^T f(x)\, e^{-i\lambda x}\, dx \right| < \varepsilon.$$

Proof: We first choose the number Λ according to Theorem I to be so large that for $|\lambda| > \Lambda$ and each $T > 1$

$$\left| \frac{1}{T} \int_0^T f(x)\, e^{-i\lambda x}\, dx \right| < \varepsilon.$$

Thus, according to Theorem II we determine a positive number $\delta = \delta(\lambda_0)$ and $T_0 = T_0(\lambda_0)$, corresponding to an arbitrary fixed λ_0 in the interval $-\Lambda \leq \lambda_0 \leq \Lambda$ such that for each λ of the interval $(\lambda_0 - \delta(\lambda_0), \lambda_0 + \delta(\lambda_0))$ and each $T > T_0(\lambda_0)$

$$\left| \frac{1}{T} \int_0^T f(x)\, e^{-i\lambda x}\, dx \right| < \varepsilon.$$

The interval $(\lambda_0 - \delta(\lambda_0), \lambda_0 + \delta(\lambda_0))$ will be denoted for brevity by $i(\lambda_0)$. Obviously because of the method of determination of this interval $i(\lambda_0)$, the conditions are satisfied for the application of the Heine-Borel covering theorem; For each point λ_0 of the closed interval $-\Lambda \leq \lambda \leq \Lambda$ is contained in $i(\lambda_0)$ (as its midpoint). By the Heine-Borel theorem it is possible to select a finite number of points of the interval $-\Lambda \leq \lambda \leq \Lambda$ e.g., $\lambda_1, \lambda_2, \ldots, \lambda_N$, such that the corresponding intervals $i(\lambda_1), i(\lambda_2), \ldots, i(\lambda_N)$ cover the complete interval $-\Lambda \leq \lambda \leq \Lambda$. Then the number

$$T_0 = \mathrm{Max}(1, T_0(\lambda_1), T_0(\lambda_2), \ldots, T_0(\lambda_N))$$

has the desired property. In fact if λ is a completely arbitrary real number then it will either be outside the interval $-\Lambda \leq \lambda \leq \Lambda$, (i.e., satisfy the inequality $|\lambda| > \Lambda$) or it will belong to at least one of the intervals $i(\lambda_1), i(\lambda_2)$,

$\ldots i(\lambda_N)$ and thus in either case the following inequality is valid: .

$$\left| \frac{1}{T} \int\limits_0^T f(x)\, e^{-i\lambda x} dx \right| < \varepsilon \ \text{ for } T > T_0.$$

82.   To bring this result into the handiest form for
the following application, we consider the purely periodic
function introduced above, namely $F_T(x)$ of period T, which
coincides with $f(x)$ in the interval $0 < x < T$. Let the

ordinary Fourier series of the function $F_T(x)$, be $\sum\limits_{-\infty}^{\infty} \alpha_n e^{i\frac{2\pi}{T}nx}$
i.e., let

$$\alpha_n = \frac{1}{T} \int\limits_0^T F_T(x)\, e^{-i\frac{2\pi}{T}nx}\, dx = \frac{1}{T} \int\limits_0^T f(x)\, e^{-i\frac{2\pi}{T}nx}\, dx.$$

From the above the following theorem follows immediately.
For each given $\varepsilon > 0$, we can determine a $T_0 = T_0(\varepsilon)$ such that
for $T > T_0$ all Fourier constants α_n of $F_T(x)$ satisfy the
inequality $|\alpha_n| < \varepsilon$; or in other words: <u>If the condition</u>
$a(\lambda) = 0$ <u>is true for all</u> λ <u>for the function</u> $f(x)$, <u>then the</u>
<u>limit equation</u>

$$\lim_{\substack{T \to \infty \\ \infty < n < \infty}} \text{upper bound } |\alpha_n| = 0.$$

<u>is true.</u>
      Thus we have proved the relation (*) of § 79 on which
the de la Vallée Poussin proof of the uniqueness theorem is
based.   From this limit equation, however, we can not con-
clude   that

$$\lim_{T \to \infty} \sum\limits_{-\infty}^{\infty} |\alpha_n|^2 = 0$$

i.e., that $\sum\limits_{-\infty}^{\infty} |\alpha_n|^2$ is small for large T, which would be
rather convenient for a direct proof of the uniqueness
theorem using the function $F_T(x)$.   But we use the simple
fact that $\sum\limits_{-\infty}^{\infty} |\alpha_n|^2$ is always <u>bounded</u>, or

$$\sum\limits_{-\infty}^{\infty} |\alpha_n|^2 = \frac{1}{T} \int\limits_0^T |F_T(x)|^2 dx = \frac{1}{T} \int\limits_0^T |f(x)|^2 dx \leqq \Gamma^2,$$

(where Γ as always, denotes the upper limit of $|f(x)|$ in $-\infty <$
$x < \infty$).   Thus we can conclude that, for example, the sum
of the <u>fourth</u> powers $\sum\limits_{-\infty}^{\infty} |\alpha_n|^4$ is small for large values of T .

In fact, we verify that

$$\sum_{-\infty}^{\infty} |\alpha_n|^4 \leq (\text{upper limit } |\alpha_n|^2) \cdot \sum_{-\infty}^{\infty} |\alpha_n|^2 \leq (\text{upper limit } |\alpha_n|)^2 \cdot \Gamma^2,$$
$$\underset{-\infty < n < \infty}{}$$

where the last quantity approaches zero as $T \to \infty$. In other words the following theorem is true: __If for the function__ $f(x)$ __the condition__ $a(\lambda) = 0$ __is satisfied for all__ λ, __then__

$$\lim_{T \to \infty} \sum_{-\infty}^{\infty} |\alpha_n|^4 = 0.$$

On this theorem we will base the following proof of the uniqueness theorem.

PROOF OF THE UNIQUENESS THEOREM

83. Let $f(x)$ be an almost periodic function, about which we shall at first make no assumption. We introduce the function $F_T(x)$ as above and consider the two folded functions:

$$g(x) = \underset{t}{M} \{ f(x+t) \overline{f(t)} \} \quad \text{and} \quad G_T(x) = \frac{1}{T} \int_0^T F_T(x+t) \overline{F_T(t)} \, dt.$$

Here, as we know, $g(x)$ is also an almost periodic function and $G_T(x)$ is a (continuous) purely periodic function of period T, whose Fourier series is given by

$$G_T(x) \sim \sum |\alpha_n|^2 e^{i \frac{2\pi}{T} nx}.$$

From this it follows that for each T

$$\frac{1}{T} \int_0^T |G_T(x)|^2 \, dx = \sum_{-\infty}^{\infty} |\alpha_n|^4.$$

The function $G_T(x)$ naturally does not need to coincide with the function $g(x)$ in the period-interval $0 < x < T$. Indeed the function does not depend in the same way on $g(x)$ as $F_T(x)$ depends on $f(x)$. This is, however, not decisive; it will be sufficient if $G_T(x)$ provides a "good approximation" to $g(x)$ in the interval $0 < x < T$ but even this is, in general, not the case for arbitrary values of T. As we know from § 53, this limit equation

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x+t) \overline{f(t)} \, dt = \underset{t}{M} \{ f(x+t) \overline{f(t)} \}$$

holds uniformly in x as $T \to \infty$. A comparison of the two functions $g(x)$ and $G_T(x)$ in the interval $0 < x < T$ amounts in essence to a comparison of the two functions

$$g_T(x) = \frac{1}{T} \int_0^T f(x+t)\, \overline{f(t)}\, dt$$

and

$$G_T(x) = \frac{1}{T} \int_0^T F_T(x+t)\, \overline{F_T(t)}\, dt \quad .$$

The two functions $g_T(x)$ and $G_T(x)$ are now by no means identical in $0 < x < T$. The equations $f(t) = F_T(t)$ and $\overline{f(t)} = \overline{F_T(t)}$, indeed, hold for the entire interval of integration $0 < t < T$ but the other factors $f(x+t)$ and $F_T(x+t)$ do not coincide in the entire interval $0 < t < T$ since $x+t$ lies partially in the interval $T < x+t < 2T$ for $0 < t < T$.
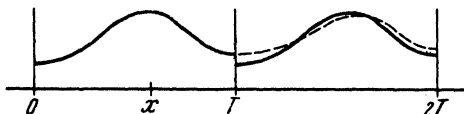

Fig. 10.

We now use our trick, which consists of taking for the "slice" T not an arbitrary number but a translation number say $T = \tau_f(\varepsilon)$. Then for $T < z < 2T$

$$\big|f(z) - F_T(z)\big| = \big|f(z) - f(z-T)\big| \leqq \varepsilon,$$

and from that the inequality

$$\big|f(x+t) - F_T(x+t)\big| \leqq \varepsilon.$$

holds for each fixed x in $0 < x < T$, and for the entire interval $0 < t < T$.

Thus we obtain (with $T = \tau_f(\varepsilon)$) the following inequality, valid for all x in $0 < x < T$

$$\big|g_T(x) - G_T(x)\big| \leqq \varepsilon \cdot \Gamma.$$

Let a sequence of numbers $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots \to 0$ be arbitrarily chosen (e.g., $\varepsilon_n = 1/n$) and let $T_1, T_2, \ldots, T_n, \ldots$ be a sequence of corresponding translation numbers (i.e., $T_n = \tau_f(\varepsilon_n)$) for which $T_n \to \infty$. Then for each value of n,

$$\big|g_{T_n}(x) - G_{T_n}(x)\big| \leqq \varepsilon_n \Gamma \quad \text{in} \quad 0 < x < T_n,$$

and thus,

$$\text{upper limit}_{0<x<T_n} |g_{T_n}(x) - G_{T_n}(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

Now, however, the sequence of functions $g_{T_n}(x)$ converges to $g(x)$ for $n \to \infty$ uniformly in $-\infty < x < \infty$. In particular it is true that

$$\text{upper limit}_{0<x<T_n} |g(x) - g_{T_n}(x)| \to 0 \quad \text{as} \quad n \to \infty$$

Thus we finally obtain as a result of the comparison of the functions $g(x)$ and $G_T(x)$, the result that <u>there exists a sequence of numbers</u> $\Gamma_1, T_2, \ldots, T_n, \ldots$ <u>of some kind with</u> $T_n \to \infty$, <u>such that</u>

$$\text{upper bound}_{0<x<T_n} |g(x) - G_{T_n}(x)| \to 0 \quad \text{as} \quad n \to \infty;$$

In other words, the function $G_{T_n}(x)$ represents in the interval $0 < x < T_n$ an ever better approximation of $g(x)$ as n becomes infinite.

From this we can conclude without difficulty that

(*) $$M\{|g(x)|^2\} = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} |G_{T_n}(x)|^2 dx \ .$$

For it is true that for a given n

$$\left| |g(x)|^2 - |G_{T_n}(x)|^2 \right| = \left| |g(x)| - |G_{T_n}(x)| \right| \cdot (|g(x)| + |G_{T_n}(x)|)$$

$$\leqq \text{upper limit}_{0<x<T_n} |g(x) - G_{T_n}(x)| \cdot 2\Gamma^2$$

in the interval $0 < x < T_n$.

Thus,

$$\left| \frac{1}{T_n} \int_0^{T_n} |g(x)|^2 dx - \frac{1}{T_n} \int_0^{T_n} |G_{T_n}(x)|^2 dx \right| \leqq \frac{1}{T_n} \int_0^{T_n} \left| |g(x)|^2 - |G_{T_n}(x)|^2 \right| dx$$

$$\leqq \text{upper limit}_{0<x<T_n} |g(x) - G_{T_n}(x)| \cdot 2\Gamma^2.$$

From this, the above limit-equation (*) follows; for the last quantity approaches zero as $n \to \infty$ and further, since the quantity $\frac{1}{T_n} \int_0^{T_n} |g(x)|^2 dx$ converges to the mean value $M\{|g(x)|^2\}$, the quantity $\frac{1}{T_n} \int_0^{T_n} |G_{T_n}(x)|^2 dx$ must approach this mean value $M\{|g(x)|^2\}$.

After this preparation it is easy to give the proof of

the uniqueness theorem. Let the hypothesis

$$a(\lambda) = M\{f(x)\,e^{-i\lambda x}\} = 0$$

be satisfied for $f(x)$ and all λ. We are to prove that $f(x)$
vanishes identically.

First, the assumption $a(\lambda) = 0$ for all λ implies, as
shown in §82, that

$$\lim_{T \to \infty} \sum_{-\infty}^{\infty} |\alpha_n|^4 = 0 \ .$$

Since now,

$$\frac{1}{T} \int_0^T |G_T(x)|^2\,dx = \sum_{-\infty}^{\infty} |\alpha_n|^4$$

for every T it follows that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |G_T(x)|^2\,dx = 0 \ .$$

With this, after what has preceded, we must conclude that

$$M\{|g(x)|^2\} = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} |G_{T_n}(x)|^2\,dx = 0 \ .$$

From the property of the almost periodic function $g(x)$ ex-
pressed in $M\{|g(x)|^2\} = 0$ it follows, however, that
$g(x)$ vanishes identically. Thus it follows, in particular,
that

$$g(0) = M_t\{f(t)\,\overline{f(t)}\} = M\{|f(t)|^2\} = 0 \ .$$

From the equation $M\{|f(t)|^2\} = 0$ it further follows that $f(x)$
itself must vanish identically, which completes the proof.

THE FUNDAMENTAL THEOREM

84. Now that the uniqueness theorem, and with it,
Parseval's equation are demonstrated, we turn to the proof
of the real <u>Fundamental Theorem</u> of the theory of almost
periodic functions: <u>The closure</u> $H\{s(x)\}$ <u>of the class of all</u>
<u>finite sums</u> $s(x) = \sum_1^N a_n e^{i\lambda_n x}$ <u>is identical with the class of</u>
<u>almost periodic functions</u>.

We already know (§ 49) that the class $H\{s(x)\}$ is con-

tained in the class of almost periodic functions. It re-
mains to be proven that conversely the class of almost per-
iodic functions is contained in the class $H\{s(x)\}$, i.e.,
that the following theorem holds.

APPROXIMATION THEOREM. <u>Every almost periodic function</u>
$f(x)$ <u>can be approximated by finite sums</u> $s(x) = \sum_{1}^{N} a_n e^{i\lambda_n x}$ <u>uni-</u>
<u>formly for</u> $-\infty < x < \infty$ <u>i.e., for each</u> $\varepsilon > 0$, <u>there exists a</u>
<u>sum</u> $s(x)$ <u>of such a nature that</u>

$$|f(x) - s(x)| \leqq \varepsilon \text{ for all } x.$$

The exponents in the approximating sums $s(x)$ can be
chosen to be precisely the <u>Fourier exponents</u> Λ_n of the
function $f(x)$.

It suffices in what follows to consider the case where
the Fourier series of $f(x)$ contains infinitely many terms,
since, otherwise, by the uniqueness theorem, the function
would itself be a finite sum $s(x)$.

If any finite sum $s(x)$ approximates the function $f(x)$
with the degree of accuracy ε i.e., $|f(x) - s(x)| \leqq \varepsilon$ for all x,
then it is clear that a "foreign" exponent (i.e., one which
is distinct from the Fourier coefficients Λ_n) can enter
only in connection with a coefficient of absolute value $\leqq \varepsilon$
since

$$|a_n| = |M\{s(x) e^{-i\lambda_n x}\}| = |M\{(s(x) - f(x)) e^{-i\lambda_n x}\}| \leqq M\{\varepsilon \cdot 1\} = \varepsilon.$$

For various applications of the approximation theorem it is
of the greatest importance that one can completely omit
such "foreign" exponents λ, just as in the case of purely
periodic functions.

On the other hand for a sufficiently fine approxima-
tion every preassigned Fourier exponent Λ_n must as a
matter of fact enter into the approximating sum $s(x)$; to be
exact, whenever the measure of accuracy ε, is less than
$|\Lambda_n|$.

85. In all our previous considerations of the Fourier
series of an almost periodic function there was no reason
for choosing any particular <u>order</u> of terms. As in the
corresponding place in the theory of Fourier series of
purely periodic functions, for the proof of the approxima-
tion theorem, we have first to discuss in detail the

question of the ordering of the terms. We shall find it closest to classical Fourier theory to consider only the Fourier exponents Λ_n (and not the coefficient A_n), in the ordering.

Since a completely arbitrary enumerable set of real numbers can occur as the set of exponents $\{\Lambda_n\}$, would at first appear to be hopeless to give a general principle that, when applied to any almost periodic function, will systematically exhaust its set of exponents. By recourse to an arithmetical point of view, however, we can do this very simply by introducing the concept of a "basis" of the set of real numbers $\{\Lambda_n\}$.

In the case of a finite set of numbers $\Lambda_1, \ldots, \Lambda_N$, the concept of a basis is a very familiar one, and indeed one understands by it a finite set of numbers $\beta_1, \ldots, \beta_M$ of such a nature that each of the given numbers Λ_n can be expressed in only one way in the form

$$\Lambda_n = g_{n,1}\beta_1 + \cdots + g_{n,M}\beta_M$$

with integral coefficients $g_{n,m}$: In the transition to an enumerable set $\Lambda_1, \Lambda_2, \ldots$ the concept of a basis must be extended since the basis must include enumerably many numbers $\beta_1, \beta_2, \ldots$ In order to be able to introduce the concept of a basis of an arbitrary enumerable set of numbers, it is necessary to permit the coefficients $g_{n,m}$ to be not only integral but also rational.

86. Let $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ be any set of enumerably many real numbers. By a basis

$$B = \{\beta_1, \beta_2, \beta_3, \ldots\}$$

we mean an at most enumerable set of real numbers β_n with the following properties:

1. The numbers $\beta_1, \beta_2, \beta_3, \ldots$ are linearly independent, i.e., no relation holds among any m of them of the following form:

$$R_1\beta_1 + R_2\beta_2 + \cdots + R_m\beta_m = 0$$

with rational coefficients $R_1, R_2, \ldots, R_m$ which do not all vanish.

2. Each of the numbers Λ_n can be represented, for sufficiently large m in the form

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m$$

with rational coefficients $r_1, r_2, \ldots, r_m$. Here it is obvious from 1., that the representation of Λ_n is unique (except, of course, for terms with zero coefficients).

Example: The set of all rational numbers $\{\Lambda_n\}$ has the basis $\{1\}$ The set $\Lambda_n = \log n$ ($n = 1, 2, 3, \ldots$) has as basis the set $\{\log p_n\}$, where p_n runs through the sequence of prime numbers $2, 3, 5, \ldots$

87. We now wish to show that <u>each sequence</u> $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ <u>has a basis</u> and indeed one whose numbers β_n all belong to the set $\{\Lambda_n\}$. Here we proceed as follows:

As the first number of our basis β_1 we take the first number Λ_{n_1} of the sequence $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$, which is not zero (thus either Λ_1 or Λ_2), and we cross out each number Λ_n of the sequence $\Lambda_{n_1+1}, \Lambda_{n_1+2}, \ldots$, for which a relation of the form $r_1\beta_1 + r_2\Lambda_n = 0$ holds in rational numbers r_1, r_2 with $r_2 \neq 0$. Let Λ_{n_2} be the first number of the sequence $\Lambda_{n_1+1}, \Lambda_{n_1+2}, \ldots$, which is not crossed out in this manner. Then we set $\beta_2 = \Lambda_{n_2}$ and cross out every number Λ_n still remaining in the sequence $\Lambda_{n_2+1}, \Lambda_{n_2+2}, \ldots$, for which a relation holds of the form $r_1\beta_1 + r_2\beta_2 + r_3\Lambda_n \neq 0$ in rational numbers r_1, r_2, r_3 with $r_3 \neq 0$. After that we set $\beta_3 = \Lambda_{n_3}$ where Λ_{n_3} denotes the first number remaining in the sequence $\Lambda_{n_2+1}, \Lambda_{n_2+2}, \ldots$ etc. Now there are two distinct possibilities. On one hand, this procedure can terminate after a finite number of steps, say p steps, so that the entire set of numbers $\Lambda_{n_p+1}, \Lambda_{n_p+2}, \ldots$ are crossed out (and thus no Λ remain). In this case the finite set

$$\{\beta_1, \beta_2, \ldots, \beta_p\} = \{\Lambda_{n_1}, \Lambda_{n_2}, \ldots, \Lambda_{n_p}\}$$

obviously forms a basis. On the other hand the procedure may never terminate. In this case a basis will obviously be given by the enumerable sequence

$$\{\beta_1, \beta_2, \beta_3, \ldots\} = \{\Lambda_{n_1}, \Lambda_{n_2}, \Lambda_{n_3}, \ldots\} .$$

88. We choose <u>any basis at all</u> $B = \{\beta_1, \beta_2, \beta_3, \ldots\}$ <u>for the sequence of exponents</u> $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ <u>of our given almost periodic function</u> $f(x)$. <u>From now on this basis will be consid-</u>

ered fixed. For the sake of convenience we wish to assume
that <u>each</u> number of the enumerable set of numbers

$$r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m$$

occurs among the exponents Λ_n (where m is arbitrary and
$r_1, r_2, \ldots, r_m$ are arbitrary rational numbers). We have only
to add the missing terms $A_n e^{i\Lambda_n x}$ with coefficients $A_n = 0$

in the series $\sum A_n e^{i\Lambda_n x}$. We carry out the proof of the
approximation theorem for the case where the basis $\{\beta_1, \beta_2, \beta_3, \ldots\}$ is enumerably infinite. The simplifications that
are intended when our basis is finite are evident. (We
could immediately go from this case to the general case by
successive addition of new numbers to extend the basis to an
infinite basis).

Now let q be an arbitrary positive integer and let Q
denote $q! = 1 \cdot 2 \cdot 3 \cdots q$. Further let P denote the number $P =
qQ (= q \cdot q!)$ so that $\frac{P}{Q} \to \infty$ as $q \to \infty$. We denote by E_q the
finite set which consists of all numbers Λ_n which can be
written in the form

$$\Lambda_n = \frac{\nu_1}{Q}\beta_1 + \frac{\nu_2}{Q}\beta_2 + \cdots + \frac{\nu_q}{Q}\beta_q$$

with the help of our basis, when $\nu_1, \nu_2, \ldots, \nu_q$ denote <u>integers</u>
all numerically $\leqq P (= qQ)$. We write an arbitrary number
Λ_n in the form

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m = \frac{s_1}{t_1}\beta_1 + \frac{s_2}{t_2}\beta_2 + \cdots + \frac{s_m}{t_m}\beta_m,$$

where we assume that the fractions $s_1/t_1, s_2/t_2, \ldots, s_m/t_m$ are re-
duced to their lowest terms and the denominators $t_1, t_2, \ldots, t_m$
are positive. Thus those numbers Λ_n surely belong to the
set E_q in case the integers $m, t_1, t_2, \ldots, t_m$ are all $\leqq q$ and the
rational numbers $r_1, r_2, \ldots, r_m$ are all numerically $\leqq q$.

Each set E_q obviously contains the preceding set E_{q-1},
i.e.,

$$E_1 \subset E_2 \subset \cdots \subset E_q \subset \cdots;$$

Further, by what we just remarked, each fixed

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m = \frac{s_1}{t_1}\beta_1 + \frac{s_2}{t_2}\beta_2 + \cdots + \frac{s_m}{t_m}\beta_m$$

is surely contained in some set E_q for sufficiently large q,
i.e., for $q \geqq q_0 = q_0(n)$, where by what precedes, we can use
for q_0 the value

$$q_0 = \text{Max}\{m, t_1, t_2, \ldots, t_m, |r_1|, |r_2|, \ldots, |r_m|\}.$$

89.   We consider now a sequence of finite sums which we write for convenience as infinite series, namely

$$s_q(x) = \sum k_n^{(q)} A_n e^{i \cdot \Lambda_n x}, \qquad q = 1, 2, 3, \ldots,$$

where the exponents Λ_n, which actually enter into the sum $s_q(x)$ (i.e., for which the factor $k_n^{(q)}$ preceding the term $A_n e^{i \cdot \Lambda_n x}$ is distinct from zero), all belong to E_q . We postulate that $0 \leqq k_n^{(q)} \leqq 1$ for all n and q, and that $k_n^{(q)} \to 1$ for n __fixed__ and $q \to \infty$.

It is easy to show with the aid of Parseval's equation, that each such sequence of finite sums $s_q(x)$ converges in the mean to $f(x)$, i.e., that

$$M\{|f(x) - s_q(x)|^2\} \to 0 \quad \text{as} \quad q \to \infty.$$

In fact from $f(x) \sim \sum A_n e^{i \cdot \Lambda_n x}$ and $\quad s_q(x) = \sum k_n^{(q)} A_n e^{i \cdot \Lambda_n x}$, it follows that

$$f(x) - s_q(x) \sim \sum (1 - k_n^{(q)}) A_n e^{i \Lambda_n x}.$$

Thus it follows from Parseval's equation that

$$M\{|f(x) - s_q(x)|^2\} = \sum_1^{\infty} (1 - k_n^{(q)})^2 |A_n|^2.$$

Now for a given $\varepsilon > 0$ let the number N be chosen so large that $\sum_{N+1}^{\infty} |A_n|^2 < \dfrac{\varepsilon}{2}$, then obviously

$$M\{|f(x) - s_q(x)|^2\} < \sum_1^{N} (1 - k_n^{(q)})^2 |A_n|^2 + \frac{\varepsilon}{2},$$

and it follows immediately, for sufficiently large values of q , that

$$M\{|f(x) - s_q(x)|^2\} < \varepsilon.$$

For by our assumption, the N numbers $k_1^{(q)}, k_2^{(q)}, \ldots, k_N^{(q)}$ all converge to 1 as $q \to \infty$.

90.   It was shown in § 73: If a sequence $f_1(x), f_2(x),$ $\ldots, f_n(x), \ldots$ of almost periodic functions converges __in the mean__ to the almost periodic function $f(x)$ then, in order to

be able to conclude that the convergence is <u>uniform</u> in $-\infty < x < \infty$ it suffices to know that the sequence $f_1(x), f_2(x),$ $\ldots, f_n(x), \ldots$ is <u>majorized</u> by the function $f(x)$. <u>The sequence of finite sums $s_q(x)$ constructed above, will, therefore, surely converge uniformly to $f(x)$ if the constants $k_n^{(q)}$ can so be chosen that the sequence of functions $s_q(x)$ is majorized by $f(x)$.</u>

     91. For this purpose we bring in again the Fejér kernel,

$$K_n(t) = \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) e^{-i\nu t}$$

which played a fundamental role in the theory of purely periodic functions, and we construct, after Bochner, the "composite" kernel

$$K^{(q)}(t) = K_P\left(\frac{\beta_1}{Q}t\right) \cdot K_P\left(\frac{\beta_2}{Q}t\right) \cdots K_P\left(\frac{\beta_q}{Q}t\right)$$

$$= \sum_{\nu_1=-P}^{P} \sum_{\nu_2=-P}^{P} \cdots \sum_{\nu_q=-P}^{P} \left(1 - \frac{|\nu_1|}{P}\right)\left(1 - \frac{|\nu_2|}{P}\right) \cdots \left(1 - \frac{|\nu_q|}{P}\right) e^{-it\left(\frac{\nu_1}{Q}\beta_1 + \frac{\nu_2}{Q}\beta_2 + \cdots + \frac{\nu_q}{Q}\beta_q\right)}.$$

Because of the linear independence of the numbers $\beta_1, \beta_2, \ldots, \beta_q$ no two terms of this finite sum can be combined, (i.e., no two contain the same factor $e^{i\lambda t}$). The numbers

$$\frac{\nu_1}{Q}\beta_1 + \frac{\nu_2}{Q}\beta_2 + \cdots + \frac{\nu_q}{Q}\beta_q,$$

which enter into the exponents, further, are exactly the numbers Λ_n which belong to the set E_q defined above.

     We write $K^{(q)}(t)$ in the form

$$K^{(q)}(t) = \sum k_n^{(q)} e^{-i\Lambda_n t}$$

(where $k_n^{(q)} = 0$ as long as Λ_n does not belong to the set E_q). Then the numbers $k_n^{(q)}$, thus defined satisfy the above conditions $0 \leqq k_n^{(q)} \leqq 1$ for all n and q, and $k_n^{(q)} \to 1$ as $q \to \infty$ for n fixed. This is easily shown. The first is evident and the second follows thus: Let n be <u>fixed</u> and let

$$\Lambda_n = r_1\beta_1 + r_2\beta_2 + \cdots + r_m\beta_m = \frac{s_1}{t_1}\beta_1 + \frac{s_2}{t_2}\beta_2 + \cdots + \frac{s_m}{t_m}\beta_m.$$

Then as was noticed (in §88) above, the number Λ_n belongs to the set E_q for $q \geqq q_0 = q_0(n)$, where

$$q_0 = \text{Max}\{m, t_1, t_2 \ldots, t_m, |r_1|, |r_2|, \ldots, |r_m|\},$$

and it is obvious that

$$\Lambda_n = \frac{\nu_1}{Q}\beta_1 + \frac{\nu_2}{Q}\beta_2 + \cdots + \frac{\nu_m}{Q}\beta_m + 0\beta_{m+1} + \cdots + 0\beta_q,$$

where $\nu_1 = r_1 Q,\, \nu_2 = r_2 Q,\, \ldots,\, \nu_m = r_m Q$. Now we have

$$k_n^{(q)} = \left(1 - \frac{|\nu_1|}{P}\right)\left(1 - \frac{|\nu_2|}{P}\right)\cdots\left(1 - \frac{|\nu_m|}{P}\right)\cdot 1 \ldots 1,$$

and since each of the m numbers

$$\frac{|\nu_1|}{P} = |r_1|\frac{Q}{P},\qquad \frac{|\nu_2|}{P} = |r_2|\frac{Q}{P},\qquad \ldots,\qquad \frac{|\nu_m|}{P} = |r_m|\frac{Q}{P}$$

approaches zero as $q \to \infty$, (since $Q/P = 1/q \to 0$ as $q \to \infty$), we have $k_n^{(q)} \to 1$ as $q \to \infty$.

92.   Now we wish to show that the sequence of functions

$$s_q(x) = \sum k_n^{(q)} A_n e^{i\Lambda_n x}$$

formed with the aid of the chosen constants $k_n^{(q)}$ __will be majorized by the function__ $f(x)$, i.e., to every arbitrarily given $\varepsilon > 0$ the corresponding translation numbers $\tau = \tau_f(\varepsilon)$ are at the same time translation numbers of each of the functions $s_q(x)$.   Then by the above proof we will have shown that this sequence of finite sums $s_q(x)$ converges __uniformly__ to $f(x)$ as $q \to \infty$.

The proof rests on the fact that for an arbitrary q

$$s_q(x) = \underset{t}{M}\{f(x + t) K^{(q)}(t)\},$$

where $K^{(q)}(t)$ denotes the above composite kernel.   This formula is very easy to establish.   In fact from

$$f(x + t) \sim \sum A_n e^{i\Lambda_n x} \cdot e^{i\Lambda_n t},$$

it follows that

$$s_q(x) = \sum k_n^{(q)} A_n e^{i\Lambda_n x} = \sum k_n^{(q)} \underset{t}{M}\{f(x + t) e^{-i\Lambda_n t}\}$$

$$= \underset{t}{M}\{f(x + t) \sum k_n^{(q)} e^{-i\Lambda_n t}\} = \underset{t}{M}\{f(x + t) K^{(q)}(t)\}.$$

__Now, however, the composite kernel__ $K^{(q)}(t)$ __possesses the same important properties as the Fejér kernel__ $K_n(t)$ __itself__; namely, first, that

$$K^{(q)}(t) \geqq 0$$

for all t, (which is evident from the definition of $K^{(q)}(t)$ as

the product of simple Fejér kernels), and secondly that

$$M\{K^{(q)}(t)\} = 1,$$

which is equally evident when one notices that the mean
value $M\{K^{(q)}(t)\}$ is equal to the constant term of $K^{(q)}(t)$ which
is exactly 1.

By application of these two properties of $K^{(q)}(t)$ it now
follows easily that each arbitrary translation number τ of
$f(x)$ corresponding to a given $\varepsilon > 0$ is at the same time a
translation number of the finite sum $s_q(x)$ corresponding to
ε . In fact we find, for an arbitrary x

$$s_q(x + \tau) - s_q(x) = M_t\{f(x + \tau + t) K^{(q)}(t)\} - M_t\{f(x + t) K^{(q)}(t)\}$$

$$= M_t\{[f(x + \tau + t) - f(x + t)] \cdot K^{(q)}(t)\}.$$

Now if $\qquad |f(x + \tau + t) - f(x + t)| \leqq \varepsilon \quad$ for all $\quad t$;

then it follows (since $K^{(q)}(t) \geqq 0$) that

$$|s_q(x + \tau) - s_q(x)| \leqq \varepsilon M_t\{K^{(q)}(t)\} = \varepsilon,$$

which shows that τ is indeed a $\tau_{s_q}(\varepsilon)$.

Thus the sequence of finite sums $s_q(x)$ converges uni-
formly to $f(x)$, as $q \to \infty$ and the proof of the fundamental
theorem is complete.

AN IMPORTANT EXAMPLE

93. As a special case of almost periodic functions we
have previously considered thus far only the purely periodic
functions of period 2π . These functions are characterized
(within the class of almost periodic functions) by the fact
that their Fourier exponents are all integral; the exponents
are, therefore, in this case rather strongly connected arith-
metically -- which gave rise to the expression that the
oscillations e^{inx} are harmonic.

In contrast to this we wish now to consider the case of
completely disharmonic oscillations, i.e., the case when the
exponents $\varLambda_n$ are linearly independent, i.e., where there
exists no linear relation whatsoever

$$g_1 \varLambda_1 + g_2 \varLambda_2 + \cdots + g_m \varLambda_m = 0$$

with integral non zero coefficients $g_1, g_2, \ldots, g_m$. In other
words the set of exponents $\{A_n\}$ forms a __basis__ here. (In
particular no A_n is 0).

This case can in a certain sense, be thought of as the
"general case", if one thinks of the Fourier exponents as
being chosen in succession in a purely random or chance
manner. In fact if the first n Fourier coefficients $A_1, A_2,$
..., A_n are fixed there are only enumerably many numbers
linearly dependent on these, while there is a non-enumerable
set of real numbers λ available from which to choose A_{n+1}

In this case of linear independence, the results are as
simple as one can imagine. In particular the following
theorem is valid:

CONVERGENCE THEOREM: __For an almost periodic function__
$f(x)$ __with linearly independent Fourier exponents the Fourier__

__series__ $\sum A_n e^{i A_n x}$ (__with arbitrary arrangement of terms__) __con-__

__verges uniformly for all__ x, __since, in fact, the series formed__
by the absolute values $\sum |A_n|$ converges.

By virtue of the uniqueness theorem we can, therefore, write
in this case:
$$f(x) = \sum A_n e^{i A_n x} .$$

For the proof of this theorem we consider again the
Fejér kernel $K_n(t)$ in the simplest case where $n = 2$ Thus,
$$K_2(t) = 1 + \tfrac{1}{2}(e^{-it} + e^{it})   (= 1 + \cos t).$$

Now let v_n be the "phase" of the nth coefficient A_n, or
$A_n = |A_n| e^{iv_n}$. We form for any value of N a "composite" ker-
nel which is, indeed, the product not of the kernels $K_2(A_n t)$,
but of the kernels resulting from the phase translation K_2
$(A_n t + v_n)$. Thus we form the kernel,
$$K_N(t) = K_2(A_1 t + v_1) \cdot K_2(A_2 t + v_2) \ldots K_2(A_N t + v_N).$$

By multiplying it out we obtain,
$$K_N(t) = 1 + \tfrac{1}{2}\{e^{-iv_1} e^{-iA_1 t} + e^{-iv_2} e^{-iA_2 t} + \cdots + e^{-iv_N} e^{-iA_N t}\} + R(t),$$

where $R(t)$ is a finite trigonometric sum $\sum a_n e^{i\lambda_n t}$ whose

exponents λ_n are (because of the linear independence of A_n)
distinct from the numbers $0, -A_1, -A_2, \ldots, -A_n, \ldots$

As in the proof of the approximation theorem, we form
the folded function $M\{f(x + t) K_N(t)\}$, but only for $x = 0$, i.e.,

we form the mean value

$$\underset{t}{M}\{f(t)\,\mathsf{K}_N(t)\}.$$

Since

$$\underset{t}{M}\{f(t)\,e^{-i\lambda t}\} = \begin{cases} A_n \text{ for } \lambda = \Lambda_n, \\ 0 \text{ for } \lambda \neq \Lambda_n \end{cases}$$

it follows that

$$\underset{t}{M}\{f(t)\,\mathsf{K}_N(t)\} = \tfrac{1}{2}\{A_1\,e^{-iv_1} + A_2\,e^{-iv_1} + \cdots + A_N\,e^{-iv_N}\}$$
$$= \tfrac{1}{2}\{|A_1| + |A_2| + \cdots + |A_N|\}.$$

But $\mathsf{K}_N(t) \geqq 0$ for all t, and $M\{\mathsf{K}_N(t)\} = 1$. Thus the left side is at most equal to Γ if we denote the upper bound of $|f(t)|$ by Γ. We thus obtain the inequality

$$|A_1| + |A_2| + \cdots + |A_N| \leqq 2\,\Gamma,$$

whereby the absolute convergence of the series $\sum A_n$ is demonstrated.

It should be noted, as a matter of orientation that in the relation $\sum|A_n| \leqq 2\Gamma$ the factor 2 has no real significance; it is there only because we used a kernel of the second order in our proof. · If we had used in place of $K_2(t)$ the kernel

$$K_p(t) = 1 + \frac{p-1}{p}(e^{-it} + e^{it}) + \cdots$$

we would have been led to the relation $\sum|A_n| \leqq \dfrac{p}{p-1}\,\Gamma$, and from that, (by taking the limit as $p \to \infty$), we should have found that $\sum|A_n| \leqq \Gamma$. <u>In this relation the equality sign alone is true</u> since obviously the upper bound of $|\sum A_n\,e^{i\Lambda_n x}|$ can not be greater than the upper bound of $\sum|A_n|$.

Because of the convergence of $\sum|A_n|$ we have just shown that

$$\underset{-\infty < x < \infty}{\text{upper limit}} |f(x)| = \sum|A_n|$$

which could have been proved from a well known theorem of Kronecker in Diophantine Approximations. This Kronecker theorem was used in the original proof of the convergence theorem.

GENERALIZATIONS OF ALMOST PERIODIC FUNCTIONS

94.   The first papers on almost periodic functions were
the basis for a number of further papers. Some of these are
on various generalizations of this class of functions.   The
first such generalizations are due to Stepanoff; independ
ently of him, Wiener in his important investigations on
Fourier integrals was also led to one of Stepanoff's classes
of functions. A very far reaching generalization was later
given by Besicovitch. Finally, Weyl gave a generalization
of the concept of almost periodicity different from the above
mentioned ones. He has investigated his generalized class
of functions by his own methods, at the end of his interest-
ing work "Integralgleichungen und fast-periodische Funk-
tioners" (Integral Equations and Almost Periodic Functions),
in which new proofs are given of the main theorems of the
ordinary theory of almost periodic functions. Some inter-
esting contributions to the generalized theory by R. Schmidt,
Kowanko, Franklin and Ursell, should also be mentioned.
        These various generalizations and others related to
them have recently been the subject of a detailed investiga-
tion by Besicovitch and the author. This brief appendix
will be concerned chiefly with the general viewpoints which
we took in that investigation in order to arrive at a uni-
fied treatment of the entire complex of questions.

        95. First of all, as an introduction to the subject,
let us say a few words on the theory "proper" of almost
periodic functions which was expounded in detail in the pre-
ceding lectures.
        By the closure of a set Φ of functions $f(x)$ $(-\infty < x < \infty)$
we mean the set $H(\Phi)$ of functions which is obtained from
the given set Φ by extending it, adding to it all those
functions which can be approximated uniformly for all x
by functions of the set Φ.
        In the following, E will throughout denote the set of

all finite sums of pure vibrations

$$s(x) = \sum_{n=1}^{N} a_n e^{i\lambda_n x}.$$

Furthermore, let F denote the set of all almost periodic functions; all continuous functions $f(x)$ with the property that for each $\varepsilon > 0$ there exists a relatively dense set of numbers (translation numbers) $\tau = \tau(\varepsilon)$ such that

$$|f(x+\tau) - f(x)| \leq \varepsilon, \qquad -\infty < x < \infty.$$

The main theorem of the theory of almost periodic functions (the approximation theorem) then states that the set F is precisely the closure of E, thus

(1) $F = H(E)$.

By means of this equation (as stressed in the lectures) our class of functions is characterized by two different types of properties: on the one hand by vibration properties as the closure of finite sums of pure vibrations, on the other hand by translation properties (structural properties) as almost periodic functions.

To every almost periodic function belongs a Fourier series $\sum A_n e^{i\lambda_n x}$ which in turn uniquely determines the function $f(x)$. Bochner's summation theorem, which extends to almost periodic functions Fejer's summation theorem for the Fourier series of purely periodic functions, furnishes an algorithm which leads from the Fourier series to uniformly approximating sums $s(x)$.

96.   The task of generalizing the theory of almost periodic functions will include above all the generalization of the approximation theorem, i.e., of the above equation (1). There one can proceed in two different ways. One may either start with the left side of the equation, i.e., try to directly generalize the definition of almost periodicity, chiefly by abandoning the requirement of continuity and dealing with functions which are only subject to such conditions as measurability or integrability in the Lebesgue sense This was the procedure of, e.g., Stepanoff, who was the first to attack the problem of generalization, and with great success.   The task in this case is to study the vibration prop-

erties of the generalized almost periodic functions defined
in this way. Or conversely, one may start with the right
side of equation (1) (that is, with the vibration properties)
by generalizing the concept of the closure of the class E ,
i.e., by replacing everywhere uniform convergence with ano-
ther concept of limit. One is here confronted with the
converse task: to investigate the corresponding generalized
almost periodic properties of the closure defined in that
manner, of the set of all finite sums. This was Besico-
vitch's procedure. In our opinion the latter point of view
is the more natural one for a systematic treatment of the
generalized theory of almost periodic functions. The prob-
lem here is quite clear and unambiguous; one considers
various limiting processes G, forms each time the corres-
ponding closure $H_G(E)$ and then seeks to characterize this
class of functions by translation properties.

97. In the following I shall for brevity's sake confine
myself to functions of class L , i.e., the class of all func-
tions $f(x)$ $(-\infty < x < \infty)$, which are Lebesgue integrable in
every finite interval. (Besicovitch and I, by the way, have
investigated all classes L^p where p is any number $\geqq 1$).
In the theory of <u>purely periodic</u> functions of class L,
considered on a finite interval (a, b) the following well
known concept of limit is adopted: $\lim f_n(x) = f(x)$ shall mean
that the mean value

$$\frac{1}{b-a}\int_a^b |f(x) - f_n(x)| \, dx$$

tends to 0 for $n \to \infty$. Or, expressed differently, we intro-
duce a notion of distance of two functions in the well known
fashion by saying that the distance between $f(x)$ and $g(x)$ is
defined by

$$D[f(x), g(x)] = \frac{1}{b-a}\int_a^b |f(x) - g(x)| \, dx$$

and then the concept of limit $\lim f_n(x) = f(x)$ is defined
simply by

$$D[f(x), f_n(x)] \to 0 \quad \text{for} \quad n \to \infty \ .$$

In the theory of almost periodic functions we always

deal with the _infinite_ interval $-\infty < x < \infty$. If we want
to extend the above concept of limit, or rather the above
concept of distance, from a finite to an infinite interval
we are faced with the choice of several different possi-
bilities each of which has its special characteristics and
its special interest. We introduce three such concepts of
distance and denote them by $D_S[f(x), g(x)]$, $D_B[f(x), g(x)]$ and
$D_W[f(x), g(x)]$ because they are very closely connected with
the generalizations of almost periodic functions by Stepan-
off, Besicovitch and Weyl.

Stepanoff's concept of distance is given by

$$D_S[f(x), g(x)] = \underset{-\infty < x < \infty}{\text{upper bound}} \frac{1}{L} \int_{x}^{x+L} |f(\xi) - g(\xi)| \, d\xi.$$

There L is a fixed positive number; its value is irrelevant
(it may be taken, e.g., $= 1$) because this concept of dis-
tance defines a concept of limit which does not depend on
the particular value of L, as can easily be seen.

With Besicovitch's concept of distance $D_B[f(x), g(x)]$ the
mean value is extended immediately over the whole interval
$-\infty < x < \infty$:

$$D_B[f(x), g(x)] = \lim_{T \to \infty} \sup \frac{1}{2T} \int_{-T}^{T} |f(x) - g(x)| \, dx.$$

Finally, Weyl's concept of distance is intermediate be-
tween the two; we here consider, to start with, a fixed
length L, as in Stepanoff's case, but let it afterward grow
without bound:

$$D_W[f(x), g(x)] = \lim_{L \to \infty} \underset{-\infty < x < \infty}{\text{upper bound}} \frac{1}{L} \int_{x}^{x+L} |f(\xi) - g(\xi)| \, d\xi.$$

It is easy to prove that the limit as $L \to \infty$ really exists.

For each of the three concepts of distance, the corres-
ponding concept of limit

$$S\text{-}\lim f_n(x) = f(x), \quad B\text{-}\lim f_n(x) = f(x), \quad W\text{-}\lim f_n(x) = f(x)$$

is defined by the relation

$$D_S[f(x), f_n(x)] \to 0, \quad D_B[f(x), f_n(x)] \to 0, \quad D_W[f(x), f_n(x)] \to 0,$$

and by using, each time, the corresponding concept of

limit, the closure of the set E is formed, that is,

$$H_S(E), \qquad H_B(E), \qquad H_W(E).$$

as the case may be.
As is easy to see, we then have

$$H_S(E) \subset H_W(E) \subset H_B(E).$$

   98. Our problem now is to characterize each of the
three classes of functions $H_S(E), H_B(E), H_W(E)$ thus de-
fined, by means of translation properties. This is the very
same problem as the one solved by equation

   (1) $H(E) = F$

for the closure $H(E)$, this closure being defined by uniform
convergence.
   By an extremely simple but fundamentally important con-
sideration, this problem can be reduced to the corresponding
problem of the original theory, which is solved by equation
(1) . By noting the fact that uniform convergence is the
narrowest of all limiting processes considered -- i.e.,
that the closure $H(E)$ obtained with uniform convergence is the
smallest of all closures $H(E)$ we can use the approximation
theorem (1) of the original theory directly as a valid re-
sult in building up the generalized theories, without having
to come back to the proof of this theorem. For, from the
inequality

$$E \subset H(E) \subset H_G(E),$$

where G denotes any one of the above three limiting processes
S-, B- or W-lim ,it follows immediately that $H_G(E) \subset H_G[H(E)]$
$\subset H_G[H_G(E)]$ and hence, using $H_G[H_G(\ldots)] = H_G(\ldots)$, that

$$H_G(E) = H_G[H(E)];$$

but because of $H(E) = F$, this means that

   (2) $H_G(E) = H_G(F)$.

This last equation however shows that the problem of char-
acterization of the class $H_G(E)$ of functions is simply equiv-
alent to investigating the effect of the respective limiting
processes G as applied to the almost periodic functions
proper of the original class F, We thus do not even have
to deal any more with the class E, i.e., with the theory of
vibrations; the connection between vibrations and translation

properties is given once and for all by the original equation (1) ; whatever remains to be investigated is concerned only with pure translation properties.

99. As to the results obtained, i.e., as to the actual characterization of the above three classes of functions $H_S(E), H_B(E), H_W(E)$ by means of translation properties, we shall confine ourselves in this short survey to the consideration of Stepanoff's class $H_S(E)$. For the other classes of functions the results are of a much more complicated nature (and the proofs are much more difficult, especially for Besicovitch's class).

By using the concept of distance $D_S[f(x), g(x)]$ introduced above, the main result of Stepanoff's theory may be stated in an exceedingly simple way, closely analogous to the main result of the theory of almost periodic functions proper as expressed by equation (1). For it is not difficult to demonstrate that this equation (1) remains valid if on both sides of the equation the primitive (and implicitly adopted) concept of distance

$$D[f(x), g(x)] = \underset{-\infty < x < \infty}{\text{upper bound}} |f(x) - g(x)|,$$

which corresponds to the concept of limit of <u>everywhere uniform convergence</u>, is replaced by the Stepanoff concept of distance $D_S[f(x), g(x)]$. Speaking precisely, we can put it this way: If we use the expression Stepanoff <u>translation number</u> associated with ε of a function $f(x)$ (of class L) for every number $\tau = \tau_f(\varepsilon)$ for which

$$D_S[f(x + \tau), f(x)] \leqq \varepsilon$$

and if we call almost periodic in Stepanoff's sense, every function $f(x)$, having for each $\varepsilon > 0$ a relatively dense set of Stepanoff translation numbers $\tau_f(\varepsilon)$, then the set F_S of all these functions is simply the above closure $H_S(E)$, i.e., we have, in complete analogy to equation (1) , the fundamental equation

$$F_S = H_S(E).$$

100. So far in our discussion of the generalized almost periodic functions we have not even mentioned the concept of the "Fourier series" of the functions in question. In the

theory of the almost periodic functions proper, that concept
formed the basis of the entire proof of the main theorem,
i.e., of the above equation (1). That a complete avoid-
ance of this concept in the preceding considerations was
possible, is linked very closely with the leading idea of
the whole investigation, this idea being to extend theorems
directly from the original theory to the generalized classes
of functions in finished form (without going into their
proofs).

However, in the further developments of the theory of
the generalized almost periodic functions, the concept of
the Fourier series regains its place, now in the most nat-
ural connection: when it is a question of finding an <u>algor-
ithm</u> which can furnish, for a given function $f(x)$ almost
periodic in the generalized sense, finite trigonometric sums
$s(x)$ which converge toward the function $f(x)$ in the sense of
convergence appropriate to the adopted concept of distance.

Fourier series are the best imaginable means for ob-
taining such an algorithm. First of all it results that
<u>with every function of any one of the generalized classes</u>
<u>of functions considered, there can actually be associated</u>

<u>a Fourier series</u> $\sum A_n e^{i \cdot \lambda_n x}$. This follows simply from the

fact that the mean value theorem of the theory of the almost
periodic functions proper can be extended to the generalized
almost periodic functions. It results further -- still
for an arbitrary function $f(x)$ of any one of the generalized
classes of functions -- that the Fejer-Bochner method of
summation achieves exactly the same result here as it does
in the theory of the almost periodic functions proper: <u>it</u>
<u>furnishes, starting from the Fourier series of the function</u>
$f(x)$ <u>finite sums</u> $s(x)$ <u>which converge toward the given</u>
<u>function</u> $f(x)$ <u>in the sense appropriate to the concept of</u>
<u>distance adopted.</u>

101. On the basis of the fact just mentioned -- namely,
that it is possible to give an algorithm which leads from the
Fourier series $\sum A_n e^{i \lambda_n x}$ of any one of the generalized almost
periodic functions $f(x)$ to approximating sums $s(x)$ -- it might
seem at first as if one should be able to infer immediately

that the function $f(x)$ is <u>uniquely</u> determined by its Fourier series in each of the generalized cases, just as in the case of the almost periodic functions proper. A uniqueness theorem in this strong form does <u>not</u>, however, hold for the generalized almost periodic functions (this is already true for the Lebesgue integrable purely periodic functions). Indeed, for the limiting processes S-lim, W-lim, B-lim (as opposed to the ordinary limiting process), <u>the limit function</u> $f(x)$ <u>of a sequence of approximating functions</u> $f_n(x)$ is not uniquely determined in the strict sense, but <u>uniquely determined only up to a "null function"</u>, i.e., a function $h(x)$ for which the respective distances $D_S[f(x), 0]$, $D_W[f(x), 0]$ and $D_B[f(x), 0]$ are equal to zero Only if such functions $f(x), g(x), \ldots,$ which differ only by null functions, are in each case considered as "identical", will the "strong" uniqueness theorem hold again.

In the case of Stepanoff's concept of distance, as is easy to see, a function $h(x)$ is a null function only if it vanishes everywhere but on a set of Lebesgue measure zero. <u>A function almost periodic in Stepanoff's sense is thus determined by its Fourier series "almost everywhere".</u> In the two other cases, the domain of null functions comprises much more, so that there holds no such strong uniqueness theorem.

102. In closing we add an important remark concerning Besicovitch's generalization of the almost periodic functions. If instead of functions of class L we consider functions of class L^2 (i.e., the class of all functions $f(x)$ which, together with their squares, are Lebesgue integrable in every finite interval), then for the Besicovitch functions the theorem analogous to the classical Riesz-Fischer theorem for purely periodic functions of class L^2 will hold. Besicovitch's theorem is that <u>the necessary and sufficient condition for an arbitrary trigonometric series</u> $\sum a_n e^{i\lambda_n x}$ <u>to be the Fourier series of a function almost periodic in Besivooitch's sense is simply that the series</u> $\sum |a_n|^2$ <u>should converge</u>. With this theorem, the generalized theory of almost periodic functions is, in a certain sense, completed.

The reader who wishes to gain a deeper knowledge of the problems discussed in this appendix is referred, e.g., to

the recent book "Almost Periodic Functions" by Besicovitch
which deals in detail with the generalizations of almost
periodic functions.

APPENDIX II

ALMOST PERIODIC FUNCTIONS OF A COMPLEX VARIABLE

103. This appendix is a brief report on the theory of analytic almost periodic functions (developed in my third paper in the Acta Math.).

We start here with a <u>complex</u> variable $z = r e^{i\theta}$ and consider the non-denumerable infinity of powers $z^\lambda = r^\lambda e^{i\lambda\theta}$ where λ goes through all real values from $-\infty$ to $+\infty$. We consider these powers on the Riemann surface $L: 0 < r < \infty$ $-\infty < \theta < \infty$ of infinitely many sheets, i.e., on the logarithmic surface with the two branch points of order infinity at $z = 0$ and $z = \infty$. Our problem then is:

<u>How must an analytic function $f(z)$, regular on a circular ring $r_1 < r < r_2$ $-\infty < \theta < \infty$ with infinitely many sheets on the surface L, be constituted in order for $f(x)$ to be representable as a sum of denumerably many powers z^{λ_n}, i.e., for $f(x)$ to admit of a meaningful representation of</u> the form

$$\sum a_n z^{\lambda_n}$$

on the entire circular ring (i.e., on all sheets at the same time, not only on every partial region made up of finitely many sheets of the ring)?

Of special interest is the case $r_1 = 0$ (or the corresponding case $r_2 = \infty$) where our problem specializes into the question: how must a function behave at a logarithmic branch point if it admits of an "irregular" Laurent series

$\sum a_n z^{\lambda_n}$ at that point?

In order to connect this problem as directly as possible with the theory of almost periodic functions as given in the preceding lectures -- which theory will form the real basis for the theory to be developed in the present case -- it will be convenient in the following to go from the Riemann surface L to the "schlicht" plane of the variable $s = \sigma + it$ by means of the simple transformation $z = e^s$ (or $s = \log z$).

The problem then reads:

Which analytic functions $f(s)$ regular in a strip $\sigma_1 < \sigma < \sigma_2$ can be developed into a series of the form

$$\sum a_n e^{\lambda_n s},$$

i.e., into a so-called "Dirichlet series"?

To the above case $r_1 = 0$ (or $r_2 = \infty$) corresponds here the case $\sigma_1 = -\infty$ (or $\sigma_2 = +\infty$) in which case the strip $\sigma_1 < \sigma < \sigma_2$ is extended to a whole left (or right, respectively) half-plane.

We wish at once to stress that our concept of a "Dirichlet series" differs from the usual one in two essential points. Firstly, we do not start with series written down in an arbitrary way, but we admit only such series as are generated by a function; secondly, the real exponents λ_n are not subject to any condition whatever in our case, whereas usually only such sequences of exponents are considered as do not have a finite point of accumulation. This second remark is of decisive importance for the whole theory, just as in the case of a real variable; only the dropping of all restrictions on the sequences of exponents makes possible a simple and easily understandable characterization of the class of functions which can be developed into Dirichlet series.

104.   As we shall see, the theory of almost periodic functions of a complex variable shares some essential features with the theory of almost periodic functions of a real variable. One part of the new theory however, follows an entirely different direction, being based on the added premise of the analytic character of the functions.

The basic definition here is the following:

An analytic function regular in a strip $\alpha < \sigma < \beta$ ($-\infty \leqq \alpha < \beta \leqq +\infty$) is called "almost periodic in (α, β)" if for every ε there exists a length $l = l(\varepsilon)$ such that every interval $a < t < a + l$ of length l on the imaginary axis contains at least one translation number $\tau = \tau(\varepsilon)$ associated with ε, i.e., a number τ satisfying the inequality

$$|f(s + i\tau) - f(s)| \leqq \varepsilon$$

for all s **in the entire strip** $\alpha < \sigma < \beta$.

The requirements made on the function are thus seen to be: firstly, on every vertical straight line $\sigma = \sigma_0$ of the strip $\alpha < \sigma < \beta$ it should be an almost periodic function of the (real) ordinate t , and secondly, the almost periodicity should take place "uniformly" on the various straight lines.

As can be shown by a somewhat complicated method of construction, there actually exist analytic functions $f(s)$ almost periodic on every straight line of a vertical strip but lacking the above-mentioned "uniform" almost periodicity on the different straight lines. A detailed investigation of these rather peculiar functions will be published soon by the Danish mathematician R. Petersen.

Every purely periodic function $f(s)$ in a strip will, of course, also be almost periodic in that strip. Hence in particular every "pure vibration" $ae^{\lambda s}$ where λ denotes an arbitrary real number, is almost periodic in the whole plane $(-\infty, +\infty)$.

It is convenient for the exposition to introduce, besides the above concept "almost periodic in (α, β)," also the concept "almost periodic in $[\alpha, \beta]$"; this to denote simply that $f(s)$ is almost periodic in the partial strip $(\alpha + \delta, \beta - \delta)$ for every $\delta > 0$.

105. The theorem that every function $f(s)$ almost periodic in (α, β) is **bounded** in $[\alpha, \beta]$ (i.e., in every partial strip) holds in the present case too, just as it holds for almost periodic functions of a real variable. The following theorem, a kind of **converse** of the preceding one, is of great importance: If we know of a function $f(s)$ regular in the strip (α, β) only that it is almost periodic on **one single line** $\sigma = \sigma_0$ of the strip (i.e., that the function $f(\sigma_0 + it) = F(t)$ is an almost periodic function of t), then $f(s)$ will be almost periodic in the whole strip $[\alpha, \beta]$, provided it is bounded in $[\alpha, \beta]$. This theorem permits the introduction of the concept of a "maximal strip" in which a function $f(s)$ is almost periodic, i.e., the largest strip $[\alpha^*, \beta^*]$, that contains the given strip $[\alpha, \beta]$ in which $f(s)$ is almost periodic and which is such that in it the function is bounded (and therefore almost periodic).

The above-mentioned theorem also furnishes us with a very convenient means for transferring complete theorems from the theory of almost periodic functions of a real variable directly to that of the complex variable, without going into the proofs a second time. Thus by means of that theorem the following can, for instance, be proved at once: The sum of two functions almost periodic in a strip $[\alpha, \beta]$, is itself almost periodic in $[\alpha, \beta]$. For $f(s)$ and $g(s)$, and hence also $f(s) + g(s)$ are bounded in $[\alpha, \beta]$; and on an arbitrarily chosen fixed straight line $\sigma = \sigma_0$, $f(\sigma_0 + it)$ and $g(\sigma_0 + it)$ and hence also $f(\sigma_0 + it) + g(\sigma_0 + it)$ are almost periodic functions of t. Also the product, and the limit function of a uniformly convergent sequence of almost periodic functions are themselves almost periodic; hence, in particular, every series of the form

$$\sum a_n e^{\lambda n s}$$

which is uniformly convergent in $[\alpha, \beta]$, represents a function almost periodic in $[\alpha, \beta]$. Furthermore -- as in the case of real variables - the theorem holds true that the integral of an almost periodic function is itself almost periodic provided merely that it stays bounded. But the two following theorems are new and without analogue in the real variable case: The derivative of an almost periodic function is itself almost periodic, and (essential for various investigations) the quotient $f(s)/g(s)$ of two almost periodic functions is always almost periodic provided only that the denominator $g(s)$ has no zeros in the strip; it is therefore, enough to assume only that $g(s) \neq 0$, instead of assuming $|g(s)| > k > 0$; in fact it is a characteristic property of an analytic almost periodic function $g(s)$ that if it is $\neq 0$ throughout (α, β), then the absolute value $|g(s)|$ must have a positive lower bound in every partial strip.

It can be shown in fact that for the almost periodicity of the quotient $h(s) = f(s)/g(s)$ of two almost periodic functions $f(s)$ and $g(s)$ it is sufficient that the function $h(s)$ be regular in the strip in question; in other words, it is quite permissible for the denominator $g(s)$ to vanish at certain points, provided only that the numerator $f(s)$ has zeros (of at least the same multiplicity) at the same points.

This theorem, which apparently forms a starting point for the development of a theory of the meromorphic almost periodic functions, can be thought of as a generalization of a beautiful theorem by Ritt on exponential polynomials.

106. In order to get to the Dirichlet expansion $\sum A_n e^{A_n s}$ of a function $f(s)$ almost periodic in (α, β), we proceed as follows. On every fixed straight line $\Re(s) = \sigma$ of the strip, $f(\sigma + it) = F_\sigma(t)$ is almost periodic in t and hence has a Fourier series

$$F_\sigma(t) \sim \sum A_n^{(\sigma)} e^{i A_n^{(\sigma)} t} .$$

Now it can be shown easily, using Cauchy's integral theorem, that in the first place the exponents $A_n^{(\sigma)}$ are independent of σ, and secondly that the coefficients $A_n^{(\sigma)}$ depend on σ in the "proper" way, i.e., $A_n^{(\sigma)} = A_n e^{A_n \sigma}$ where A_n is a constant not depending on σ. The Fourier series on the non-denumerably infinity of straight lines $\Re(s) = \sigma$ thus combine into a single expansion of the form

$$\sum A_n e^{A_n \sigma} \cdot e^{i A_n t} = \sum A_n e^{A_n s}$$

and it is this expansion which we call the "Dirichlet series" of the function $f(s)$ in the strip (α, β). We write

$$f(s) \sim \sum A_n e^{A_n s} .$$

In the special case of a purely periodic function of period $2\pi i$ (or, if we return to the z-plane, of a function in the schlicht circular ring $r_1 < |z| < r_2$) our Dirichlet series becomes the ordinary Laurent series of the function $f(s)$.

107. Two theorems that carry over immediately into the present case from the case of a real variable are the uniqueness theorem which states that two different functions almost periodic on the same strip have associated with them two different Dirichlet series, and the theorem on convergence in the mean, i.e., the Parseval equation

$$M\{|f(\sigma + it)|^2\} = \sum |A_n|^2 e^{2 A_n \sigma}, \qquad \alpha < \sigma < \beta .$$

Also the rules of calculation for operations with the series expansions are extended immediately: The Dirichlet

expansion of a sum, of a product, of a limit function, of an integral (if it stays bounded) and of a derivative is obtained by formal calculation from the Dirichlet series of the given functions.

The approximation theorem also carries over without difficulty and yields the following main theorem of the whole theory: For a function analytic in $[\alpha, \beta]$ to be almost periodic, it is necessary and sufficient that it can be approximated by means of exponential polynomials

$$\sum_{1}^{N} a_n e^{\lambda_n s} \quad \text{uniformly in } [\alpha, \beta].$$ Approximating exponential polynomials can be found again by Bochner's summation method.

Examples of Dirichlet series which converge uniformly in the strip $[\alpha, \beta]$, are furnished by the case of linearly independent exponents. Further simple examples are obtained by means of the following theorem which acts as a bridge to the theory of ordinary Dirichlet series whose sequences of exponents have no finite point of accumulation: If the series

$$\sum e^{-|\Lambda_n|\delta}$$

converges for every $\delta > 0$, then the Dirichlet expansion $\sum \Lambda_n e^{\Lambda_n s}$ is absolutely convergent in the whole strip (α, β) of almost periodicity. The ordinary Laurent expansion $\sum a_n e^{n\frac{2\pi}{p}s}$, of purely periodic functions comes under this as a special case.

108. For ordinary Laurent series $\sum_{-\infty}^{\infty} a_n z^n$ the special case where all exponents are non-negative, i.e., the case of an ordinary power series $\sum_{0}^{\infty} a_n z^n$, plays a special role. In the same way for the present theory, a special interest attaches to the case where the Dirichlet exponents Λ_n of our almost periodic function $f(s)$ are all of the same sign. We therefore proceed to the consideration of such functions.

One arrives here at results which are almost as

rounded out as those obtained for the classical special case

of the proper power series $\sum_0^\infty a_n z^n = \sum_0^\infty a_n e^{ns}$ (only that con-

vergence in the usual sense in our general case need not
of course occur). We begin with an important <u>boundary value
theorem</u> the proof of which is based on the above approxima-
tion theorem and uses the classical Phragmen-Lindelof prin-
ciple.

Let

$$F(t) \sim \sum A_n e^{i A_n t}$$

be an entirely arbitrary almost periodic function (not even
assumed to be analytic) with all its <u>exponents A_n positive</u>.
Then it is the <u>boundary function</u> of an analytic function
regular in the half-plane $\sigma < 0$, i.e., there exists a func-
tion $f(s)$, regular for $\sigma < 0$ and continuous for $\sigma \leqq 0$ which on
the imaginary axis $\sigma = 0$ satisfies the boundary condition

$$f(it) = F(t)$$

and this function $f(s)$ is almost periodic in $(-\infty, 0)$, tends to
0, uniformly in t for $\sigma \to -\infty$, and its Dirichlet expan-
sion is obtained from the Fourier expansion for $F(t)$, by re-
placing it by $\sigma + it = s$, i.e.,

$$f(s) \sim \sum A_n e^{A_n s} .$$

From this boundary value theorem follows a result on almost
periodic functions of a real variable, which is surprising
at first sight: such a function is <u>completely determined by
its values on an arbitrarily small interval if its Fourier
exponents are all positive</u> (or all negative).

From the above boundary value theorem we can derive
at once the following important theorem.

<u>If a function</u> $f(s) \sim \sum A_n e^{A_n s}$ <u>almost periodic in</u> $[\alpha, \beta]$

<u>has its Dirichlet exponents all positive, then it can be
continued analytically over the whole left half-plane</u> $\sigma \leqq \alpha$
<u>and then represents a function</u> $f(s)$. <u>almost periodic in</u> $(-\infty, \beta]$
<u>which tends to</u> 0 <u>uniformly in</u> t <u>for</u> $\sigma \to -\infty$.

Indeed, we have only to consider the almost periodic
function $F(t) = f(\sigma_0 + it)$ for a fixed σ_0 in the interval

$\alpha < \sigma < \beta$.   Its Fourier series is given by $\sum A_n e^{A_n \sigma_0} e^{i A_n t}$
and has, therefore, none but positive exponents; hence it is
the boundary function of an almost periodic function analy-
tic in the whole left half-plane $\sigma < \sigma_0$. But then it must
coincide with the given function $f(s)$, because it coincides
with that function on the line $\sigma = \sigma_0$.

Corollary: Every function $f(s) \sim \sum A_n e^{A_n s}$ which is al-
most periodic in a strip (α, β) and which has its exponents
A_n bounded, $|A_n| < K$, must be an entire transcendental func-
tion. For $f(s) e^{Ks}$ has none but positive exponents and can,
therefore, be continued toward the left, while $f(s) e^{-Ks}$
has none but negative exponents and can, therefore, be con-
tinued toward the right. Such functions with bounded ex-
ponents (but which may be everywhere dense in a finite in-
terval) play a role in our theory similar to that played by
polynomials in the theory of analytic functions in a schlicht
circular ring.

If the assumption of "none but positive" exponents is
weakened to only non-negative exponents, i.e., if a constant
term c is admitted in the expansion $f(s) \sim \sum A_n e^{A_n s}$, then the
above theorem will of course hold just the same, except that
$f(s)$ will converge for $\sigma \to -\infty$, to just that constant term
(instead of zero). On the other hand, one sees immediately
that a function $f(s)$ almost periodic in a left half-plane
$[-\infty, \beta]$ can not even remain bounded for $\sigma \to -\infty$ if its
Dirichlet expansion contains at least one negative expon-
ent A_n ; for with fixed σ we have

$$\underset{-\infty < t < \infty}{\text{upper bound}} |f(\sigma + it)| \geqq |M\{f(\sigma + it) e^{-i A_n t}\}| = |A_n e^{A_n \sigma}|,$$

where the right side tends to ∞ for $\sigma \to -\infty$, because of
$A_n < 0$. Hence we can conclude: If a function almost
periodic in $[-\infty, \beta]$ remains bounded for $\sigma \to -\infty$, then it
must have only non-negative exponents and must, therefore,
tend to a definite limit.

109. It follows from the above considerations that for
any function $f(s)$ almost periodic in a half-plane $[-\infty, \beta]$

there are only three possibilities in approaching the "point"
at infinity $\sigma = -\infty$, just as is the case, in accordance
with Weierstrass theorem, for a purely periodic function
(that is, for a function regular in the neighborhood of a
point of the schlicht plane, if we return to the z-plane):

   A. $f(s)$ converges to a finite limit (case of regu-
        larity).
   B. $|f(s)|$ tends to ∞ (case of a pole).
   C.   The values of $f(s)$ lie everywhere dense in every
        half-plane $\sigma < \sigma_0$ (case of an essential singular-
        ity.)

     Indeed, we can conclude the following, literally as in
the proof of Weierstrass' theorem: If $f(s)$ does not come
under case C , i.e., if there exists a number a with
$|f(s) - a| > k > 0$, then we must have either case A or case ·
B because the function $g(s) = 1/(f(s) - a)$ is almost periodic
and bounded in a left half-plane, and therefore must tend
to a limit, in accordance with the above. If this limit
is $\neq 0$, we have case A; if it is $= 0$, we have case B.

     In case C even Picard's theorem holds, i.e., the
function $f(s)$ assumes all values with the exception of one
at most, in every left half-plane.

     But how can we recognize from the series expansion of
the function whether we have case A, B or C? The answer is
very simple (the proof however is not too easy):

   a.   The regular behavior A corresponds to the case
        where all Λ_n are $\geqq 0$·
   b.   The polar behavior B corresponds to the case in
        which negative exponents occur, amongst them a num-
        erically largest one.
   c.   The essentially singular behavior C corresponds to
        the case in which there occur negative exponents
        without there being a numerically largest one
        among these.

     Thus the last case comprises the functions whose ex-
ponents have $-\infty$ as a point of accumulation, as well as
those functions for which the exponents have a finite neg-
ative number Λ^* as lowest point of accumulation that does
not belong itself, however, to the set of the Λ_n.. A more
refined analysis however, shows up differences between these
two classes of functions; thus it can be shown that for a

function $f(s)$ of the second class there exists no exceptional value in the sense of Picard's theorem at all, i.e., that $f(s)$ assumes all values in every left half-plane.

110.   After this discussion of almost periodic functions that are defined over an entire half-plane, we return again to the consideration of almost periodic functions defined on an arbitrary vertical strip. Our aim is now to deal with the question of the <u>Laurent separation of almost periodic functions</u>. The classical Laurent theorem for a function analytic in a <u>schlicht</u> circular ring $r_1 < |z| < r_2$ or in our terminology, for a function $f(s)$, purely periodic in a strip $\sigma_1 < \sigma < \sigma_2$, states that $f(s)$ can be decomposed into two purely periodic summands

$$f(s) = f_1(s) + f_2(s)$$

such that $f_1(s)$ is analytic in the left half-plane $\sigma < \sigma_2$, and $f_2(s)$ is analytic in the right half-plane $\sigma > \sigma_1$, and such that $f_1(s)$ and $f_2(s)$ respectively are still regular at the "points" $\sigma = -\infty$ or $\sigma = +\infty$ respectively. After the above results on almost periodic functions it might seem at first as if the Laurent separation could be immediately transferred to an arbitrary function $f(s) \sim \sum A_n e^{A_n s}$ almost periodic in a strip $\sigma_1 < \sigma < \sigma_2$, by simply forming the two partial series

$$\sum_{A_n > 0} A_n e^{A_n s} \quad \text{and} \quad \sum_{A_n < 0} A_n e^{A_n s}$$

with only positive or only negative exponents respectively. These partial series would then, one might think, represent almost periodic functions $f_1(s)$ and $f_2(s)$ of which $f_1(s)$ could be continued analytically toward the left and $f_2(s)$ toward the right. But this method is not permissible offhand because it is not clear whether or not the two partial series

$$\sum_{A_n > 0} \quad \text{and} \quad \sum_{A_n < 0}$$

are, each by itself, Dirichlet expansions of almost periodic functions. And indeed it is not only this idea for the proof but even the Laurent separation itself which may fail. It can be shown by means of suitably con-

structed examples that there exists, indeed, almost periodic
functions for which a Laurent separation is not possible.
This would at first seem rather discouraging, because, prim-
arily a Laurent expansion would provide a powerful tool for
further advances in the theory, besides being of interest in
its own right. Fortunately however, a slight change of dir-
ection of thought can overcome this difficulty. For it can
be shown that the derivative $f'(s)$ of an almost periodic func-
tion $f(s)$ always admits of a Laurent expansion, even if it
itself does not. The analytic reason for this lies in the
Cauchy representation

$$f'(s) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-s)^2}\,dz$$

where the second power in the denominator makes it possible
to extend the path of integration to infinity; the corres-
ponding result is in general not possible for the repres-
entation of $f(s)$ itself,

$$f(s) = \frac{1}{2\pi i}\int \frac{f(z)}{z-s}\,dz \ .$$

It is also easily understandable from a purely formal view-
point, a superficial consideration of the Dirichlet expan-
sions of $f(s)$ and $f'(s)$, that $f'(s)$ behaves much more nicely at
$\lambda = 0$ with respect to a separation than does $f(s)$ itself.

For in passing from the series expansion $\sum A_n e^{\Lambda_n s}$ for $f(s)$

to the expansion $\sum A_n \Lambda_n e^{\Lambda_n s}$ for $f'(s)$, the terms $A_n e^{\Lambda_n s}$
acquire factors Λ_n which are very small just in the neighbor-
hood of $\lambda = 0$.

    By means of the Laurent separation of the derivative
one can in many cases actually obtain the same results as
would be·furnished by a separation (not always possible) of
$f(s)$ itself. We cite e.g., the <u>extended uniqueness theorem</u>
which states: If $f(s)$ and $g(s)$ are almost periodic in the
strips (α, β) and (γ, δ) respectively, and if the Dirichlet
expansion of $f(s)$ in (α, β) is formally identical with that of
$g(s)$ in (γ, δ), then $f(s)$ and $g(s)$ are the same analytic func-
tion, even in the case where the two strips are <u>entirely</u>
<u>outside</u> of each other, say $\alpha < \beta < \gamma < \delta$. There exists

then a function almost periodic in the entire strip $\alpha < \sigma \prec \delta$
which coincides with $f(s)$ in (α, β) and with $g(s)$ in (γ, δ). The
proof is quite analogous to the proof of the corresponding
theorem for ordinary Laurent series; only in our case there
is first established the identity of the derivatives $f'(s)$
and $g'(s)$ by means of Laurent separation, and after that we
go over to the functions $f(s)$ and $g(s)$ themselves through inte-
gration. (A more direct proof incidentally, was later dis-
covered of this extended existence theorem).

111. We consider again the case where $f(s)$ is given in
an entire half-plane. We shall discuss the so-called <u>inver-
sion theorem</u> of the theory which is of particular interest
because it expresses in a peculiar way the apparent "com-
pleteness" of the class of almost periodic functions. In
order to bring out distinctly the analogy with the corres-
ponding theorem for the special case of an ordinary power
series (and with the case, seemingly, but not really, more
general, of an algebraic singularity), we prefer to operate
in the z-plane instead of the s-plane (i.e., to perform the
transformation of variables $s = \log z$ mentioned at the begin-
ning). We then have to deal with a function $\varphi(z) = f(\log z)$ --
instead of a function $f(s)$ defined in a left half-plane --
which is analytic in a certain circular neighborhood of the
branch point $z = 0$ of infinite order and is also almost per-
iodic there with respect to the angles θ. Its Dirichlet
expansion -- or perhaps it is better here to speak of a
"generalized Laurent expansion" -- is denoted by

$$\sum A_n z^{\Lambda_n}$$

We assume that <u>the exponents Λ_n are all positive</u> (regular
case), and besides, that there is <u>a smallest</u> among them.
It can then be easily seen that our function $w = \varphi(z)$ maps a
sufficiently small neighborhood of the branch point $z = 0$ of
infinite order onto a certain full neighborhood of the
branch point $w = 0$ of infinite order. Thus we can speak of
the <u>inverse function</u> $z = \psi(w)$ of $w = \varphi(z)$ in a neighborhood
of this latter branch point $w = 0$. It can be shown further-
more -- and this is the inversion theorem -- that <u>this in-
verse function $z = \psi(w)$ is itself an almost periodic function</u>.
From this theorem it follows e.g., that the inverse

function of Riemann's zeta function $\quad \zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s} \quad$ as well as
of other important number-theoretic functions are themselves
almost periodic functions (after a harmless transformation
of variables is applied to them). It might be an interest-
ing task to investigate in more detail these almost periodic
inverse functions and in particular, their Dirichlet expan-
sions.

  112. I conclude this short sketch of the theory of al-
most periodic functions of a complex variable with a dis-
cussion of some interesting and still unpublished results
of Jessen, concerning the <u>distribution of values</u> of an al-
most periodic function. We will use again the language of
the s-plane and consider any function $f(s)$ almost periodic
in a strip $[\alpha, \beta]$. Let a be a fixed value which is assumed
by $f(s)$ in this strip. We want to know the distribution of
all a-places in our strip. Without less of generality we
may take $a = 0$ so that we deal just with the <u>zeros</u> of $f(s)$.
Let (γ, δ) be a partial strip of (α, β) which contains at least
one of the zeros of $f(s)$ in its interior. We denote by $N(T)$
the number of zeros in that part of the strip (γ, δ) whose
ordinates lie between $-T$ and T. It follows without diffi-
culty from the almost periodicity of $f(s)$ that this number
$N(T)$ grows to infinity for $T \to \infty$, and just as in the
purely periodic case it does so with order T in the sense
that the quotient $N(T)/2T$ remains between two positive con-
stants for sufficiently large values of T. There arises
quite naturally the further question as to whether the dis-
tribution of zeros in (γ, δ) is also (as in the purely periodic
case) such a regular one that the above quotient $N(T)/2T$
even tends to a definite limit $G = G(\gamma, \delta)$ for $T \to \infty$. In
that case, one could again speak of a certain "relative
frequency" of the zeros in (γ, δ). The answer is that for a
given almost periodic function $f(s)$ this is not always but
"almost always" the case, in the sense that there are <u>at
most denumerably many pairs of values</u> γ, δ <u>in the interval</u>
$\alpha < \sigma < \beta$ <u>for which the limit</u> $G(\gamma, \delta)$ <u>does not exist.</u> The
critical exceptional values of γ and δ that may exist, as
well as the limit $G(\gamma, \delta)$ for all other pairs of values γ, δ
can be characterized in the following simple fashion, ac-

cording to Jessen: For every fixed σ of the interval
$\alpha < \sigma < \beta$ one forms the mean value

$$\varphi(\sigma) = M_t\{\log|f(\sigma + it|\},$$

whose existence may be deduced from the almost periodicity
of $f(s)$. The real function $\varphi(\sigma)$ (on $\alpha < \sigma < \beta$) associated
with our analytic function $f(s)$ turns out to be continuous
and convex so that the curve given by $\varphi = \varphi(\sigma)$ possesses a
definite tangent except for at most denumerably many points
where it has corners. This convex curve $\varphi = \varphi(\sigma)$ governs
the distribution of zeros of $f(s)$ in the following simple
manner: the abcissas σ corresponding to the corners of the
curve are exactly the above-mentioned possible exceptional
values of γ and δ, whereas for every other pair of values
$\gamma, \delta,$ the derivatives $\varphi'(\gamma)$ and $\varphi'(\delta)$ both exist; so, then,
does the above limit $G(\gamma, \delta)$ which is easily determined from
the formula

$$G(\gamma, \delta) = \frac{1}{2\pi}\{\varphi'(\delta) - \varphi'(\gamma)\}.$$

On closer examination this theorem turns out to be a
natural generalization to almost periodic functions of a
classical formula by Jensen. It settles in a general way
the problem of the distribution of values of almost periodic
functions. But when it is a question of investigating the
distribution of values for a definite given almost periodic
function, and in particular, of determining more definitely
the above limit $G(\gamma, \delta)$, one must usually resort to the Dir-
ichlet expansion of the function. For the case of Riemann's
zeta function and of some related functions there exist for
a number of years detailed investigations of this kind
(which, incidentally, have recently been much refined by
Jessen and the author). We shall, however not here enter
more closely upon these rather special investigations, which
were the starting point of the whole theory of almost per
iodic functions.

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